Applied Finite Mathematics **Name: ___Solutions_________________**
Voting Methods Graded Assignment

1. Give an example of an election with four candidates satisfying the following:

   a. No candidate has a majority of the first-place votes.
   b. One candidate is a Condorcet candidate but has no first-place votes.
   c. A different candidate is the winner under the Borda Count Method.
   d. Yet a different candidate is the winner under the Plurality Method.

   Also show that your election satisfies these criteria.

   Solution:

   One example is as follows:

# of Voters	**8**	**4**	**3**	**2**
1st choice	A	B	B	D
2nd choice	C	D	C	C
3rd choice	B	C	D	B
4th choice	D	A	A	A

a. A majority of the first-place votes would be a total of 9, which no player has.

b. C beats every other candidate in a head-to-head contest, so **C** is a Condorcet candidate. However, C does not have any first-place votes.

c. Under the Borda Count Method, we have the following point totals:

$A: 4(8) + 3(0) + 2(0) + 1(9) = 41$
$B: 4(7) + 3(0) + 2(10) + 1(0) = 48$
$C: 4(0) + 3(13) + 2(4) + 1(0) = 47$
$D: 4(2) + 3(4) + 2(3) + 1(8) = 34$

Hence, **B** wins under this method.

d. Candidate A has 8 first-place votes, B has 7, C has 0, and D has 2. So, **A** wins under the Plurality method.

2. Consider the following fairness criterion: *If a majority of the voters have candidate X ranked last, then candidate X should not be a winner of the election.*

a. Give an example to show that the Plurality Method violates this criterion.

# of Voters	**14**	**10**	**8**	**4**	**1**
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

In this example, Candidate A has a majority of the last-place votes, but they win by the Plurality Method.

b. Give an example to show that the Plurality-with-Elimination Method violates this criterion.

# of Voters	**2**	**2**	**3**
1st choice	A	B	C
2nd choice	B	A	A
3rd choice	C	C	B

In this example, Candidate C wins by Plurality-with-Elimination but has a majority of the last-place votes.

c. Explain why the Method of Pairwise Comparisons satisfies this criterion.

Under the Method of Pairwise Comparisons, if a candidate has a majority of the last-place votes, then they will never win a head-to-head comparison with any other candidate. Therefore, this method satisfies the criterion described.

d. Explain why the Borda Count Method satisfies this criterion.

Consider an election with N candidates where n voters place candidate X last, and m is the rest of the voters. We know, then, that $n > m$, since a majority of the voters place X last. So, the maximum number of points that candidate X can receive is $n + Nm$, because this would be the case where the m voters place candidate X first.

Each voter gives out $1 + 2 + \cdots + N = \frac{N(N+1)}{2}$ points under this method, so the total number of points given out is $\frac{N(N+1)}{2}(n+m)$. Now, some candidate must earn $1/N^{th}$ of the total points; so, that candidate must earn $\frac{\frac{N(N+1)}{2}(n+m)}{N} = \frac{(N+1)(n+m)}{2} = \frac{Nn+Nm+n+m}{2} = \frac{(n+Nm)+(m+Nn)}{2}$. Now, since $n > m$, we know that $\frac{(n+Nm)+(m+Nn)}{2} > \frac{(n+Nm)+(n+Nm)}{2} = \frac{2(n+Nm)}{2} = n + Nm$. Therefore, some candidate will earn more points than candidate X, causing candidate to lose the election.

3. Consider a variation of the Borda Count Method in which a first-place vote in an election with *N* candidates is worth *F* points, where $F > N$, and all other places in the ballot are the same as in the ordinary Borda Count: $N - 1$ points for 2nd place, $N - 2$ points for 3rd, etc. By choosing *F* large enough, we can make this variation of the Borda Count Method satisfy the majority criterion. Find the smallest value of *F* (expressed in terms of *N*) for which this happens, and prove why your value is, in fact, the smallest value such that this happens.

Solution:

Suppose there are *k* voters and *N* candidates. We will consider two cases: when *k* is even and when it is odd.

***k* is odd:** Let $k = 2x + 1$, where $x > 0$. Suppose candidate *X* is a candidate with a majority of the first-place votes. The minimum Borda points this candidate can have is $F(x+1)+x$, since this would be the case where candidate *X* receives *F* first-place votes and every other voter places them last. The maximum Borda points any other candidate can earn is $(N-1)(x+1)+Fx$, since this would be the case where *x* voters place *X* first and the rest $(x + 1)$ of the voters place them second. So, the Majority Criterion will be satisfied when $F(x+1)+x > (N-1)(x+1)+Fx$. This means that $F > N(x+1)-(2x+1) = N(k-x)-k$.

***k* is even:** Let $k = 2x$, where $x > 0$. Suppose candidate *X* is a candidate with a majority of the first-place votes. The minimum Borda points this candidate can have is $F(x+1)+x-1$, since this would be the case where candidate *X* receives *F* first-place votes and every other voter places them last. The maximum Borda points any other candidate can earn is $(N-1)(x+1)+F(x-1)$, since this would be the case where *x* voters place *X* first and the rest of the voters place them second. So, the Majority Criterion will be satisfied when $F(x+1)+(x-1) > (N-1)(x+1)+F(x-1)$. This means that $F > \frac{N(x+1)-2x}{2} = \frac{N(x+1)-k}{2}$.