Unit 5 Lesson 6 Homework Set Solutions

1. Compute the steady-state matrix of each of the given matrices without raising them to a high power.

a. $\begin{bmatrix} .55 & .45 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$

b. $\begin{bmatrix} 3/5 & 2/5 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$

c. $\begin{bmatrix} 1 & 0 & 0 \\ .2 & .4 & .4 \\ .3 & .2 & .5 \end{bmatrix}$ $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix}$

d. $\begin{bmatrix} 1/5 & 0 & 4/5 \\ 0 & 1 & 0 \\ 0 & 3/8 & 5/8 \end{bmatrix}$ $\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix}$

e. $\begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

f. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/8 & 1/8 & 1/4 & 1/2 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{.43} & \mathbf{0} & \mathbf{0} & \mathbf{.57} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

g. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/4 & 1/4 & 1/2 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \end{bmatrix}$ $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{.5} & \mathbf{.5} & \mathbf{0} & \mathbf{0} \\ \mathbf{.5} & \mathbf{.5} & \mathbf{0} & \mathbf{0} \end{bmatrix}$

h. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ .2 & .4 & 0 & .4 \\ .1 & .2 & .4 & .3 \end{bmatrix}$ $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{.33} & \mathbf{.67} & \mathbf{0} & \mathbf{0} \\ \mathbf{.33} & \mathbf{.67} & \mathbf{0} & \mathbf{0} \end{bmatrix}$

2. David is going to risk $2 in the following game. He places a $1 bet on each repeated play of the game in which the probability of his winning $1 is 0.4, and he continues to play until he has accumulated a total of $3 or he has lost all of his money. Write the transition matrix associated with this chain, and find the probability that David will accumulate $3.

$$\boldsymbol{T} = \begin{array}{c} \\ \$0 \\ \$1 \\ \$2 \\ \$3 \end{array} \begin{array}{c} \begin{array}{cccc} \$0 & \$1 & \$2 & \$3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ .6 & 0 & .4 & 0 \\ 0 & .6 & 0 & .4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \quad \textbf{\textit{In canonical form}}: \begin{array}{c} \\ \$0 \\ \$3 \\ \$1 \\ \$2 \end{array} \begin{array}{c} \begin{array}{cccc} \$0 & \$3 & \$1 & \$2 \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline .6 & 0 & 0 & .4 \\ 0 & .4 & .6 & 0 \end{array}\right] \end{array}$$

$$\boldsymbol{R} = \begin{bmatrix} .6 & 0 \\ 0 & .4 \end{bmatrix} \quad \boldsymbol{Q} = \begin{bmatrix} 0 & .4 \\ .6 & 0 \end{bmatrix}$$

$$\boldsymbol{N} = (\boldsymbol{I} - \boldsymbol{Q})^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & .4 \\ .6 & 0 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 1.32 & 0.53 \\ 0.79 & 1.32 \end{bmatrix}$$

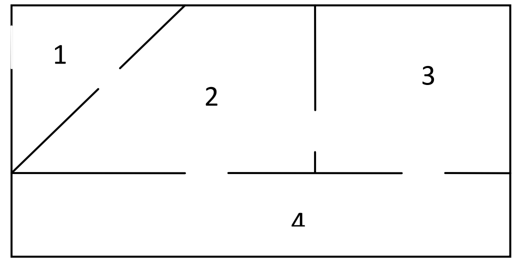
$$\boldsymbol{NR} = \begin{bmatrix} 1.32 & 0.53 \\ 0.79 & 1.32 \end{bmatrix} \begin{bmatrix} 0.6 & 0 \\ 0 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.79 & 0.21 \\ 0.47 & 0.53 \end{bmatrix}$$

$$\rightarrow \textbf{\textit{Stable} – \textit{state matrix}}: \begin{array}{c} \\ \$0 \\ \$3 \\ \$1 \\ \$2 \end{array} \begin{array}{c} \begin{array}{cccc} \$0 & \$3 & \$1 & \$2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ .79 & .21 & 0 & 0 \\ .47 & .53 & 0 & 0 \end{bmatrix} \end{array}$$

$$\rightarrow \textbf{P(David accumulates \$3 if he risks \$2)} = \mathbf{0.53}$$

(Adapted from Finite Mathematics byTan, Thomson Learning, 2003, p. 533)

3. A mouse is put into the maze in the following figure. During each time period it chooses at random one of the doors in the room it is in and moves to the next room and always leaves the current room if possible. From room 1 it can escape outside but in room 3 there is a Harahart mousetrap (the mouse is captured, but not harmed).



a. Set up the process as a Markov Chain.

$$T = \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{matrix}$$

In canonical form:

$$\begin{matrix} \\ 1 \\ 3 \\ 2 \\ 4 \end{matrix} \begin{matrix} \begin{matrix} 1 & 3 & 2 & 4 \end{matrix} \\ \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}\right] \end{matrix}$$

b. Find N and NR.

$$R = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} \end{bmatrix} \quad Q = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & 0 \end{bmatrix}$$

$$N = (I - Q)^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & 0 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1.2 & 0.4 \\ 0.6 & 1.2 \end{bmatrix}$$

$$NR = \begin{bmatrix} 1.2 & 0.4 \\ 0.6 & 1.2 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0.72 & 0.16 \\ 0.36 & 0.48 \end{bmatrix}$$

c. If the mouse starts in room 4, what is the average number of rooms he will visit before he escapes or is caught?

$$N_{21} + N_{22} = 0.6 + 1.2 = 1.8 \ \textit{rooms}$$

d. If he starts in room 4, what is the average number of times he visits room 1?

$$N_{21} = 0.6 \ \textit{times}$$

e If he starts in room 2, what is the probability he escapes?

$$\textit{Stable} - \textit{state matrix}: \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ .72 & .16 & 0 & 0 \\ .36 & .48 & 0 & 0 \end{bmatrix} \end{matrix} \rightarrow \text{P(starts in room 2 and escapes)} = 0.72$$

4. The victims of a certain disease being treated at Wake Medical Center are classified annually as follows: cured, in temporary remission, sick, or dead from the disease. Once a victim is cured, he is permanently immune. Each year, those in remission get sick again with probability 1/2 and are cured with probability 1/2, while those who are sick, are cured, go into remission, or die from the disease with probability 1/3 each.

a. Find the transition matrix.

$$T = \begin{matrix} \\ c \\ r \\ s \\ d \end{matrix}\begin{matrix} \begin{matrix} c & r & s & d \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

$$\textit{Canonical form} \quad \begin{matrix} \\ c \\ d \\ r \\ s \end{matrix}\begin{matrix} \begin{matrix} c & d & r & s \end{matrix} \\ \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{array}\right] \end{matrix} = T_C$$

b. If a victim is now in remission, find the probability he is still alive in two years.

$$T_C^2 = \begin{matrix} \\ c \\ r \\ s \\ d \end{matrix}\begin{matrix} \begin{matrix} c & r & s & d \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{2}{3} & \frac{1}{6} & \frac{1}{6} & 0 \\ \frac{1}{2} & \frac{1}{3} & 0 & \frac{1}{6} \end{bmatrix} \end{matrix}$$

$$\text{P(now in remission, alive in two years)} = \frac{2}{3} + \frac{1}{6} = \frac{5}{6} = 0.8333\ldots$$

c. If a victim is sick, what is the probability that he will eventually be cured?

$$R = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \quad Q = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix}$$

$$N = (I - Q)^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & 0 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 1.2 & 0.6 \\ 0.4 & 1.2 \end{bmatrix}$$

$$NR = \begin{bmatrix} 1.2 & 0.6 \\ 0.4 & 1.2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix}$$

$$\rightarrow \textbf{\textit{Stable} − \textit{state matrix}}: \begin{array}{c} \\ \$0 \\ \$3 \\ \$1 \\ \$2 \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{c} & \mathbf{r} & \mathbf{s} & \mathbf{d} \end{array} \\ \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0.8} & \mathbf{0.2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0.6} & \mathbf{0.4} & \mathbf{0} & \mathbf{0} \end{bmatrix} \end{array}$$

$$\rightarrow \mathbf{P(victim\ sick, will\ be\ cured) = 0.6}$$

d. If a victim is now in remission, about how long will it take before he is cured or dies from the disease?

$$\boldsymbol{N_{11} + N_{12} = 1.2 + 0.6 = 1.8\ \ years}$$