Unit 5 Lesson 5 Homework Set Solutions

1. A rat is placed in the maze shown in the figure below. During a fixed time interval, the rat randomly chooses one of the doors available to it (depending upon which room it is in) and moves through that door to the next room – it does not remain in the room it occupies. Each movement of the rat is taken as a transition in a Markov chain in which a state is identified with the room the rat is in. The first row of the transition matrix is:

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ 1 & \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} \end{array}$$

   a. Construct the entire transition matrix for this process.
   b. If the rat starts in room 1, what is the probability that it is in room 3 after two transitions? After three transitions?
   c. Determine the stable state vector
   d. After a large number of transitions, what is the probability that the rat is in room 4?
   e. In the long run, what percentage of the time will the rat spend in rooms 2 or 3?

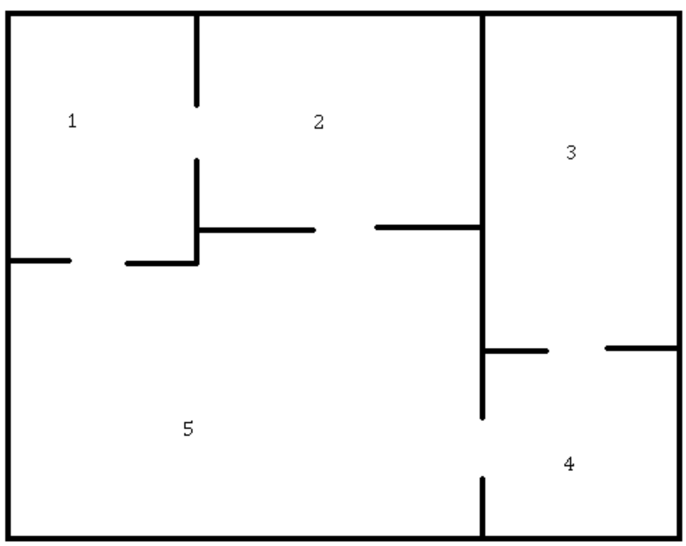

Figure: Maze for Problem 1

Problems taken from <u>Matrices</u> by the NCSSM Mathematics Department, NCTM, 1998

a.

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 \end{bmatrix} \end{array}$$

b. 0; 0.083

c. The stable state vector is

$$\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ [0.2 \quad 0.2 \quad 0.1 \quad 0.2 \quad 0.3] \end{array}$$

d. The long-run probability of being in room 4 is 0.2

e. In the long run, the rat will spend 30% of the time in rooms 2 or 3.

2. Suppose jar A contains 3 beads and jar B contains 4 beads. Of the 7 beads, 3 are red and 4 are black. We start a Markov process by picking at random one bead from each jar and interchanging them (that is, the bead from jar A is placed in jar B, and the bead from jar B is placed in jar A). This process is continued. Let the state of the process be identified by the number of red beads in jar A. State 0 represents 0 red beads in jar A, state 1 represents 1 red bead in jar A, state 2 represents 2 red beads in jar A, and state 3 represents 3 red beads in jar A.

   a. Find the transition matrix for this process, focusing on the number of red beads in jar A.
   b. If we start with the 3 red beads in A and the 4 black beads in B, what is the probability that there will be 2 red beads in jar A after 3 transitions?
   c. In the long run, what is the probability that there will be 2 red beads in jar A?

a.

$$\begin{array}{c} \\ 0\\1\\2\\3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & \frac{5}{12} & \frac{1}{12} \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

b. Starting in state 3 (all red in jar A), the probability that there will be 2 red beads in jar A after 3 transitions is 0.4236.

c. The probability that there will be 2 red beads in jar A in the long run is 0.343

3. Wade, Donald, and Andrea are playing Frisbee. Wade always throws to Donald, Donald always throws to Andrea, but Andrea is equally likely to throw to Wade or Donald.

   a. Represent this information as a transition matrix of a Markov Chain.
   b. Notice that this transition matrix has zero entries in several places. Compare the values of the transition matrix if raised to the second, fourth, sixth, and tenth powers. Are the zero entries still there? Can you explain?

a.

$$\begin{array}{c} \\ W \\ D \\ A \end{array} \begin{array}{c} \begin{array}{ccc} W & D & A \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \end{bmatrix} \end{array}$$

b.

$$p^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix} \quad p^4 = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.25 & 0.25 & 0.5 \\ 0.25 & 0.5 & 0.25 \end{bmatrix}$$

$$p^6 = \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.125 & 0.375 & 0.5 \\ 0.25 & 0.375 & 0.375 \end{bmatrix} \quad p^{10} = \begin{bmatrix} 0.1875 & 0.375 & 0.4375 \\ 0.21875 & 0.40625 & 0.375 \\ 0.1875 & 0.40625 & 0.40625 \end{bmatrix}$$

The zero entries vanish because by the sixth throw everyone has a chance to catch the frisbee no matter who has it initially.

4. The snack bar at school sells three items that students especially like: onion rings, french fries, and chocolate chip cookies. The manager noticed that what each student ordered depended on what he or she ordered on the last previous visit. She ran a survey during the first two weeks of school and found out that 50% of those who ordered onion rings on their last snack break ordered them again this time, while 35% switched to French fries and 15% switched to chocolate chip cookies. 40% of those who ordered French fries on their last visit did so the next time, but 30% switched to onion rings, and another 30% switched to chocolate chip cookies. Of the students who ordered chocolate chip cookies on their last visit, 20% switched to onion rings, and 55% switched to French fries.

   a. Set up the transition matrix for this Markov process.
   b. On Monday, 30 students buy French fries, 40 buy onion rings, and 25 buy chocolate chip cookies. If these same students come in on Tuesday and each buys one of these items, how many orders of French fires should the manager expect to sell?
   c. Suppose the students in part 4b continue buying from the snack bar every day for two weeks. How many orders of onion rings, French fries, and cookies should the manager expect to sell on the third Monday?
   d. If these same people come all year, how many orders of onion rings, French fries, and cookies should the manager expect to sell to them each day?
   e. In the long run, what percent of the orders for these three items will be onion rings? French fries? Chocolate chip cookies?

a.

$$\begin{array}{cc} & \begin{array}{ccc} \boldsymbol{OR} & \boldsymbol{FF} & \boldsymbol{CC} \end{array} \\ \begin{array}{c} \boldsymbol{OR} \\ \boldsymbol{FF} \\ \boldsymbol{CC} \end{array} & \begin{bmatrix} \mathbf{0.5} & \mathbf{0.35} & \mathbf{0.15} \\ \mathbf{0.3} & \mathbf{0.4} & \mathbf{0.3} \\ \mathbf{0.2} & \mathbf{0.55} & \mathbf{0.25} \end{bmatrix} \end{array}$$

b. The manager should expect to sell $\mathbf{40(0.35) + 30(0.4) + 25(0.55) = 39.75}$***, which rounds to 40 orders of French fries.***

c. Assuming 5 days in a week, this will involve 10 transitions.

$$[\mathbf{40 \quad 30 \quad 25}]\,\boldsymbol{T}^{\mathbf{10}} = [\mathbf{32.8 \quad 39.7 \quad 22.45}]$$

The manager will sell 33 orders of onion rings, 40 orders of fries, and 22 cookies.

d. Assume 180 school days in the year.

$$[\mathbf{40 \quad 30 \quad 25}]\,\boldsymbol{T}^{\mathbf{180}} = [\mathbf{32.82 \quad 39.73 \quad 22.45}]$$

The manager will sell 33 orders of onion rings, 40 orders of fries, and 22 cookies each day.

e. The rows of the powers of the transition matrix converge to values that indicate that the onion rings will be ordered 34.5% of the time, French fries will be ordered 41.8% of the time, and cookies will be ordered 23.6% of the time.

5. The manager of the snack bar in the previous problem decided to add soft custard ice-cream cones to the menu. Lots of students tried the new item, but few of them liked it. (The machine didn't work right and the ice cream came out lumpy.) A two-week survey gave the results in the transition matrix below.

$$\begin{matrix} Rings \\ Fries \\ Cookies \\ Cones \end{matrix} \begin{pmatrix} Rings & Fries & Cookies & Cones \\ 0.4 & 0.3 & 0.1 & 0.2 \\ 0.25 & 0.35 & 0.2 & 0.2 \\ 0.15 & 0.4 & 0.2 & 0.25 \\ 0.3 & 0.35 & 0.3 & 0.05 \end{pmatrix}$$

   a. Does this system reach a stable state?
   b. If everyone really dislikes the ice cream, why does the stable state matrix show that many students still buy it?
   c. Why is a Markov chain not a good model for this system? Are people likely to forget that they did not like something only two days after they ate it? (Recall the principal assumption of a Markov process.) Why would this model work if the customers liked the ice cream?

a. Yes; the stable state vector is

$$[\mathbf{0.2825} \quad \mathbf{0.3455} \quad \mathbf{0.1900} \quad \mathbf{0.1822}]$$

b. The stable state vector shows that 18.22% of the students buy ice cream. A Markov Chain model assumes that each step in the process depends only on what happened in the previous step, so it is assumed that people who had ice cream more than 1 day before forgot that they did not like it.

c. A Markov chain is not a good model for the situation. People will remember that they do not like the ice cream, so what they buy each day is dependent on more than what they bought the day before.

6. A mouse is in the maze shown in the figure below. Doors are shown by openings between rooms. Arrows indicate one-way doors and the direction of passage through the one way doors. The mouse does not have to change rooms at each transition, but can stay in a room. Notice some of the rooms are impossible to leave once they are entered. During each transition, the mouse has an equal chance of leaving a room by a particular door or staying in the room. For example, during a single transition, a mouse in room 2 has a 1 in 6 chance of moving to room 1, a 1 in 6 chance of staying in room 2, a 1 in 3 chance of moving to room 3, and a 1 in 3 chance of moving to room 4.

   a. Find the transition matrix which describes the movement of the mouse.
   b. If the mouse starts in room 4, what is the probability that it will eventually be trapped in room 1?
   c. In which room besides 7 should the mouse be started to have the best chance of being trapped in room 7? Besides 1, the worst chance?

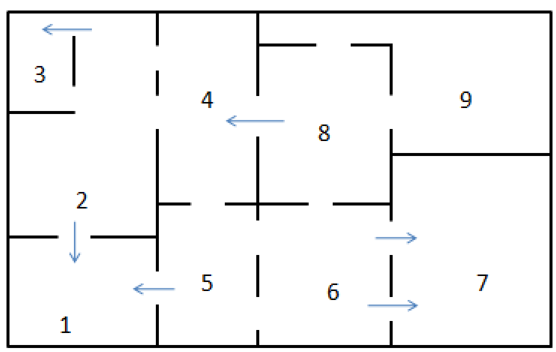

Figure: Maze for Problem 6

a.

	1	2	3	4	5	6	7	8	9
1	1	0	0	0	0	0	0	0	0
2	1/6	1/6	1/3	1/3	0	0	0	0	0
3	0	1/2	1/2	0	0	0	0	0	0
4	0	1/2	0	1/4	1/4	0	0	0	0
5	1/5	0	0	1/5	1/5	2/5	0	0	0
6	0	0	0	0	1/3	1/6	1/3	1/6	0
7	0	0	0	0	0	0	1	0	0
8	0	0	0	1/5	0	1/5	0	1/5	2/5
9	0	0	0	0	0	0	0	2/3	1/3

b. 0.7757

c. Starting in room 6 offers the highest probability (0.6355) of ending in Room 7. Starting in either room 2 or 3 offers the same low probability (0.1495) of ending in Room 7.

7. Every census gives statisticians many new ideas to investigate. In the 1980 census a question was asked about the occupations of the fathers surveyed. Of the fathers who were professionals, 63% of their sons were professionals and 20% of their sons were in service oriented jobs. Among fathers in service-oriented jobs, 31% of their sons were professionals and 45% worked in service jobs. Among fathers who worked in manufacturing, only 18% of their sons followed their profession and 41% went into service-oriented jobs.

   a. Based on the 1980 census, if John's great-grandfather was in a service-oriented job, what is the probability that John will also be in a service-oriented job?
   b. Suppose the 1980 census showed that 32% of the men were in manufacturing, 41% were professionals, and 27% were in service jobs. In ten generations, what percentage of the population of descendants will be professionals?

The transition matrix for this Markov chain is

$$\begin{array}{c} \\ \boldsymbol{Prof.} \\ \boldsymbol{Serv.} \\ \boldsymbol{Menu.} \end{array} \begin{array}{c} \begin{array}{ccc} \boldsymbol{Prof.} & \boldsymbol{Serv.} & \boldsymbol{Manu.} \end{array} \\ \begin{bmatrix} \mathbf{0.63} & \mathbf{0.20} & \mathbf{0.17} \\ \mathbf{0.31} & \mathbf{0.45} & \mathbf{0.24} \\ \mathbf{0.41} & \mathbf{0.41} & \mathbf{0.18} \end{bmatrix} \end{array}$$

a. 0.3335

b.

$$[\mathbf{0.41} \quad \mathbf{0.27} \quad \mathbf{0.32}]\,\boldsymbol{T}^{\mathbf{10}} = [\mathbf{0.484} \quad \mathbf{0.321} \quad \mathbf{0.194}]$$

In ten generations, 48% will be professionals.

8. A dreaded strain of flu is studied by research biologists. Statistics are taken each week in an effort to describe the probabilities after exposure of staying well, getting ill, becoming immune, and dying. A person becomes immune to this flu by having a mild case of it. Any well person once exposed to this flu has a 20% chance of getting the illness. Once a person becomes ill there is a 55% chance of remaining ill for more than a week, a 40% chance of being permanently immune after a mild illness, and a 5% chance of dying from the illness.

   a. Construct a transition matrix to represent the information above.
   b. Compute the probability of becoming immune after ten weeks for groups that begin as follows:
      i. 100% well and exposed
      ii. 50% well and 50% sick
      iii. 80% well and 20% immune
      iv. 100% sick
   c. Does this Markov process have a stable state?

a.

$$\begin{array}{c} \\ \textbf{Well} \\ \textbf{Ill} \\ \textbf{Immune} \\ \textbf{Dead} \end{array} \begin{array}{c} \begin{array}{cccc} \textbf{Well} & \textbf{Ill} & \textbf{Immune} & \textbf{Dead} \end{array} \\ \begin{pmatrix} 0.80 & 0.20 & 0 & 0 \\ 0 & 0.55 & 0.40 & 0.05 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

b. (i): 0.7189 (ii): 0.8028 (iii): 0.7751 (iv): 0.8866

c. The Markov chain does not have a stable state because of the absorbing states; however, the transition matrix raised to a large power does stabilize as

$$\begin{array}{c} \\ \textbf{Well} \\ \textbf{Ill} \\ \textbf{Immune} \\ \textbf{Dead} \end{array} \begin{array}{c} \begin{array}{cccc} \textbf{Well} & \textbf{Ill} & \textbf{Immune} & \textbf{Dead} \end{array} \\ \begin{pmatrix} 0 & 0 & 0.8889 & 0.1111 \\ 0 & 0 & 0.8889 & 0.1111 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$