Complex Numbers
Algebra 2

Goals:

1. Operate with numbers to solve problems. (1.01)
   d. Define complex numbers and perform basic operations with them.

Materials and Equipment Needed:

1. Copy of Student Handout for each student (There is no follow-up problem.)
2. calculator

Activity 1: Definitions, Basic Operations (addition, subtraction, multiplication) and Graphing

Define imaginary numbers: For any positive real number, b, $\sqrt{-b^2} = \sqrt{b^2} * \sqrt{-1}$ or bi where i is the imaginary unit and bi is called a pure imaginary number.

If $\sqrt{-1} = i$ then what is i^2?

$$i^2 = (\sqrt{-1})^2 = -1$$
$$i^3 = i^2 * i = -1 * i = -i$$
$$i^4 = i^2 * i^2 = -1 * -1 = 1$$
$$i^5 = i^4 * i = 1 * i = i$$
$$i^6 = i^5 * i = i * i = i^2 = -1$$

Notice the pattern? $i,\ -1,\ -i,\ 1,\ i,\ -1,\ \ldots$

Define complex numbers: A complex number is one of the form $a+bi$, where a and b are real numbers and i is the imaginary unit; a is called the real part and b the imaginary part.

How do we add and subtract complex numbers? You can think of it in a similar way to adding and subtracting expressions with constants and variables: add "like terms". For example,
$(4+6i)+(2-5i) = 4+2+6i-5i = 6-i$

Notice your calculator has a "complex mode". If you press the MODE key, on the seventh line you will see an option $a+bi$. You should select it. This will allow you to check your work on the calculator.

How do we multiply complex numbers? Again, you can think of combining "like terms". This will combine real parts with real parts and imaginary parts with imaginary parts. Then you go back to the definition of imaginary numbers. Here are several examples:

- $7i * 3i = 21 * i^2 = 21 * (-1) = -21$
- $\sqrt{-12} * \sqrt{-5} = \sqrt{-4*3} * \sqrt{-5} = 2i\sqrt{3} * i\sqrt{5} = 2i^2\sqrt{3*5} = 2(-1)\sqrt{15} = -2\sqrt{15}$
- $(7+5i)(-2) = -14-10i$
- $(4+6i)(2-i) = 8-4i+12i-6i^2 = 8+8i-6(-1) = 8+6-8i = 14+8i = 2(7+4i)$

Here are two examples taken from the Algebra 2 Indicators.

- Solve for a and b if $(a+bi)^2 = 5+12i$

Begin by expanding $(a+bi)^2 = a^2+2abi+b^2i^2 = a^2+2abi-b^2$. Gathering the real and imaginary parts together we have $a^2-b^2+2abi = 5+12i$. If we match up real and imaginary coefficients, we see

$a^2 - b^2 = 5$ and $2ab = 12$. Solving the second expression for a and substituting into the first expression we have $a = \frac{12}{2b} = \frac{6}{b}$ and then $(\frac{6}{b})^2 - b^2 = 5$ or $\frac{36}{b^2} - b^2 = 5$. By looking at a graph of $\frac{36}{b^2} - b^2 = 5$ we can find the solutions to be $b = \pm 2$. If $b = 2$ then $a = \frac{6}{2} = 3$ and if $b = -2$ then $a = \frac{6}{-2} = -3$. Let's check our answers to make sure (and to get a little more practice).

$$(3+2i)^2 = (3+2i)(3+2i) = 9+6i+6i+4i^2 = 9+12i-4 = 5+12i$$

and

$$(-3-2i)^2 = (-3-2i)(-3-2i) = 9+6i+6i+4i^2 = 9+12i-4 = 5+12i$$

- Simplify $\frac{3+4i}{2-i} + \frac{6-5i}{2+2i}$

Use the product of the two denominators to get the common denominator. Then you will be able to combine terms.

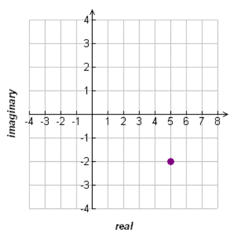
$$\frac{3+4i}{2-i} + \frac{6-5i}{2+2i} = \frac{(3+4i)(2+2i)}{(2-i)(2+2i)} + \frac{(6-5i)(2-i)}{(2-i)(2+2i)} = \frac{(6+6i+8i+8i^2)+(12-6i-10i+5i^2)}{(2-i)(2+2i)}$$

$$\frac{(6+14i-8)+(12-16i-5)}{4+4i-2i-2i^2} = \frac{(-2+14i)+(7-16i)}{4+2i+2} = \frac{-2+7+14i-16i}{6+2i} = \frac{5-2i}{4+2i}$$

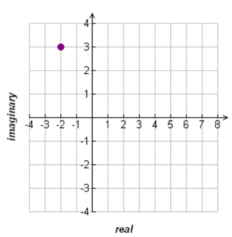
Note the calculator gives you a different answer if you try to check this one. It is also doing the division in the last step.

Graphing:

Graphing complex numbers is much like plotting points. Instead of having an x-axis and a y-axis, you have a real axis (horizontal) and an imaginary axis (vertical). To graph a number like $5-2i$ you would go right 5 in the real direction and down 2 in the imaginary direction. You can see this point plotted in the graph below.



How would we graph $(-2+3i)$?



Note your calculator will not graph complex numbers in their complex form.

Activity 2: Julia Sets: An application using complex numbers and iteration

You have become familiar with many real valued functions such as $f(x)=x$ or $g(x)=x^2$. We can also have complex valued functions that allow complex numbers both as input and output values. Suppose we begin by investigating a simple complex function, called $f(z)=z^2$. We use the variable z to represent a complex number just like we used x to represent a real number in the function $g(x)=x^2$. Let's look at some input and output values of our function $f(z)=z^2$:

z	$f(z)$
$0+0i$	$(0+0i)^2=0$
$-1+0.3i$	$(-1+0.3i)^2=0.91-0.6i$
$(-0.25+.5i)$	$(-0.25+.5i)^2=-0.1875-.25i$
$0+i$	$(0+i)^2=-1$

You are probably familiar with iteration on your calculator, although you may not have known it by its name. Here is an example of iteration: Suppose you input the number 2 in your calculator and press the ENTER key. Now if you press the x^2 key, the calculator will square 2 and result in 4. If you press the x^2 key again, the calculator will square 4 and result in 16. Every time you press the x^2 key, the calculator will square the number it had as its previous result. You are repeatedly putting a value into a function, getting a value out and using that resulting value as a new input. This is the basic idea of iteration.

Suppose you iterated the function $f(z)=z^2$ using $z=0.4003125+0.4725i$ as the initial input value. What happens?

$$f(0.4003125+0.4725i)\doteq -0.0630061523+0.3782953125i$$
$$f(-0.0630061523+0.3782953125i)\doteq -0.1391375682-0.476698642i$$
$$f(-0.1391375682-0.476698642i)\doteq 0.0170868469+0.13265338i$$
$$f(0.0170868469+0.13265338i)\doteq 1.159911472*10^{-4}+4.533255987*10^{-4}i$$
$$f(1.159911472*10^{-4}+4.533255987*10^{-4}i)\doteq -1.920501522*10^{-7}+1.051635125*10^{-7}i$$
$$f(-1.920501522*10^{-7}+1.051635125*10^{-7}i)\doteq 2.582389659*10^{-14}-4.039333716*10^{-14}i$$
$$f(2.582389659*10^{-14}-4.039333716*10^{-14}i)\doteq -9.647480513*10^{-28}-2.086226723*10^{-27}i$$
$$f(-9.647480513*10^{-28}-2.086226723*10^{-27}i)\doteq -3.421603139*10^{-54}+4.025366332*10^{-54}i$$
$$f(-3.421603139*10^{-54}+4.025366332*10^{-54}i)\doteq 0$$

In words what we did was:

- Start with the complex number $z=0.4003125+0.4725i$
- Iterate repeatedly using the function $f(z)=z^2$
- Stop when the result *stabilizes*, in this case at zero $f(z)=z^2$

Now we need to investigate what happens to our function $f(z)=z^2$ if we iterate it using lots of different starting values. (Note we are also asked for the distance of z from the origin in our table. The calculator can easily find this. Press the MATH key. Arrow to the right two times until the cursor is on CPX. Select Option 5:abs(. This will take you back to the home screen. Enter the value you are interested in and close the parentheses. This will give you this distance your point is from the origin. For example, $abs(1+0.5i)=1.118033989$. This tells us the complex number $1+0.5i$ is approximately a distance of 1.118033989 from the origin.) It might be more efficient to divide the task of filling in this chart among the students. It is not crucial that each student investigate all values.

	Z - values for iterating $f(z) = z^2$	where does it stabilize?	what is the distance of z from the origin?
1.	$-0.1771875+0.49875i$	0	0.5292891201
2.	$-0.984375+0.0196875i$	0	0.9845718553
3.	$0+i$	1	1
4.	$0.5446875-0.511875i$	0	0.7474627003
5.	$-1.424065+0.15751i$	Overflow	1.432749289
6.	$0.6759375-0.7021875i$	0	0.9746582935
7.	$-1.0434375+0.3609375i$	Overflow	1.104100401
8.	$-0.879375+0.63i$	Overflow	1.081758009
9.	$-0.5184375-1.1615625i$	Overflow	1.272008209
10.	$-0.8925-0.66218125i$	Overflow	1.111323651
11.	$0+0.0853125i$	0	0.0853125
12.	$0.4725+1.273125i$	Overflow	1.35797773
13.	$1.614375+1.2009375i$	Overflow	2.012077911
14.	$0.0525-0.8465625i$	0	0.8481888448
15.	$-0.60375+0.3478125i$	0	0.6967694006
16.	$0.76125+0.7284375i$	Overflow	1.05362363
17.	$1.44375+0i$	Overflow	1.44375
18.	$1.3059375-0.905625i$	Overflow	1.58922289
19.	$-0.7021875-0.55125i$	0	0.8927171151
20.	$0.4003125+0.4725i$	0	0.6192788933
21.	$0-i$	1	1
22.	$0.2690625-0.170625i$	0	0.3186024475
23.	$0.9646875+0.07875i$	0	0.9678964486
24.	$0.3478125-1.05i$	Overflow	1.10610738
25.	$-1+0i$	1	1

26.	$-1.44375-0.6890625i$	Overflow	1.599756604
27.	$-1.0565625-0.0721875i$	Overflow	1.059025661
28.	$0.380625+0.5775i$	0	0.6916513866
29.	$0.0853125+0.2296875i$	0	0.2450195305

Having filled in the chart, do you see patterns between the second two columns? When the iteration takes you to zero, the distance from the origin is between 0 and 1. When the iteration takes you to 1, the distance from the origin is 1. When the iteration results in calculator overflow, the distance from the origin is greater than 1.

If we were able to plot with a black dot all the complex numbers in the world that, upon iteration, stabilize *somewhere*, in this case 0 or 1, they would form a black circle centered at the origin with radius one. (look at PowerPoint slide 1) You will notice there are other colors on the graph. Points that are colored something other than black resulted in overflow or did not stabilize. The colors represent how quickly the overflow occurred.

This doesn't seem particularly interesting and, in fact, seems like a lot of work for not much reward. However, if we change the function slightly, we will see some very interesting results☺ Rather than using the function $f(z)=z^2$, let's look at $f(z)=z^2+c$, where c represents some complex number.

Suppose we use a c-value of $c=-0.12+0.74i$ What would you expect to see? (slide 2) All the points colored black are initial values that, when iterated, stabilized somewhere. The points that are any other color did not stabilize when iterated.

Here are some other interesting c-values:

$c=-0.481762-0.531657i$ (slide 3)
$c=-0.15652-1.03225i$ (slide 4)
$c=0.11031-0.67037i$ (slide 5)

These are called *fractals*. More specifically, they are known as Julia Sets, named for the French mathematician Gaston Julia (1893-1978). (slide 6 is a picture of G. Julia – he lost his nose while serving in WWII)

If you then make a graph of all the c-values from the Julia Sets that are connected, you have the Mandelbrot Set (slide 7), named for the mathematician Benoit Mandelbrot. Mandelbrot was born in Poland in 1924 and his family immigrated to France in 1936. (slide 8 is a picture of B. Mandelbrot – he is still an active mathematician, often talking with large groups of mathematics teachers at national conferences)

Student Handout
Complex Numbers
Algebra 2

1. From our definition we know $\sqrt{-1} = i$. What is i^2?

$$i^3 =$$
$$i^4 =$$
$$i^5 =$$
$$i^6 =$$

What pattern do you notice in your results?

2. Simplify each of the expressions. Try them first without your calculator, then check them using the complex number feature of your calculator.

a. $(4+6i)+(2-5i)=$

b. $7i*3i=$

c. $\sqrt{-12}*\sqrt{-5}=$

d. $(7+5i)(-2)=$

e. $(4+6i)(2-i)=$

3. a. Solve for a and b if $(a+bi)^2 = 5+12i$.

b. Simplify $\dfrac{3+4i}{2-i}+\dfrac{6-5i}{2+2i}$.

(both from the Algebra 2 Indicators)

4. Graph the complex numbers $5-2i$ and $(-2+3i)$.

5. Iterate $f(z) = z^2$ for the following Z–values. Determine where it stabilizes and distance from the origin.

	Z - values for iterating $f(z) = z^2$	where does it stabilize?	what is the distance of z from the origin?
1.	$-0.1771875 + 0.49875i$		
2.	$-0.984375 + 0.0196875i$		
3.	$0 + i$		
4.	$0.5446875 - 0.511875i$		
5.	$-1.424065 + 0.15751i$		
6.	$0.6759375 - 0.7021875i$		
7.	$-1.0434375 + 0.3609375i$		
8.	$-0.879375 + 0.63i$		
9.	$-0.5184375 - 1.1615625i$		
10.	$-0.8925 - 0.66218125i$		
11.	$0 + 0.0853125i$		
12.	$0.4725 + 1.273125i$		
13.	$1.614375 + 1.2009375i$		
14.	$0.0525 - 0.8465625i$		
15.	$-0.60375 + 0.3478125i$		
16.	$0.76125 + 0.7284375i$		
17.	$1.44375 + 0i$		
18.	$1.3059375 - 0.905625i$		
19.	$-0.7021875 - 0.55125i$		
20.	$0.4003125 + 0.4725i$		
21.	$0 - i$		
22.	$0.2690625 - 0.170625i$		
23.	$0.9646875 + 0.07875i$		
24.	$0.3478125 - 1.05i$		
25.	$-1 + 0i$		
26.	$-1.44375 - 0.6890625i$		
27.	$-1.0565625 - 0.0721875i$		
28.	$0.380625 + 0.5775i$		
29.	$0.0853125 + 0.2296875i$		