

Paper Folding: Maximizing the Area of a Triangle Algebra 2

(This lesson was developed by Jan Baysden of Hoggard High School and Julie Fonvielle of Whiteville High School during the Leading to Success in Algebra 2 Workshop in June 2003.)

Algebra 2 Goals

- Operation with algebraic expression to solve problems. (1.02)
- Describe graphically, algebraically, and verbally real-world phenomena as functions; identify the independent and dependent variables. (3.01)
- Find and interpret the maximum of a function. (3.06)
- Use equations which contain radical expressions to solve problems. Solve by graphing. (3.11)
- Write and interpret an equation of a curve which models data. (4.01)

Materials Needed:

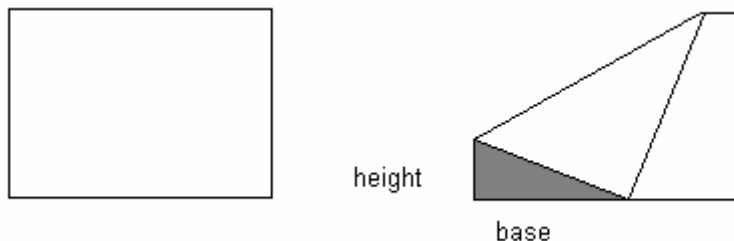
- Copy of student handout for each student
- Two sheets of 8.5 by 11 inch paper for each student
- Ruler for each student
- Graphing calculator

This paper folding activity will have two parts: one activity with the paper in landscape (long side horizontally) position and the other with the paper in portrait (long side is vertical) position. Although the process is the same for both, the domain and results will be different. After the initial activity, watch students to be sure they are folding the paper in the right position.

Activity One: Folding Warm-Up

Given a sheet of 8.5 by 11 inch paper have students fold a triangle. Be careful to watch that they place the top left corner along the bottom edge of the paper. Give them some time to experiment to see the sizes of the possible triangles. Have students find the area—this will give them some time to remember how to measure and how to calculate area before they begin the main part of the problem. Students can compare results and record the dimensions of the triangle with maximum area from the ones folded. This will allow comparison once they have completed the lesson.

Activity Two: Folding Triangles (Landscape)



Students will orient the paper in landscape position and fold the top left corner to fall along the bottom edge. The triangle of interest is shaded above. Because the top left corner will reach the bottom edge in a number of ways, students will measure the height and base and calculate the area for a group of different triangles. As the students go through this process, we hope they will notice that:

- height has limitations in its size--from 4.25 inches to 0 inches
- every triangle is a right triangle
- sum of the height and the hypotenuse is 8.5 inches.

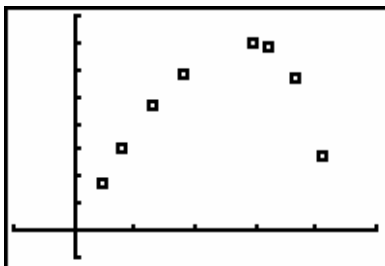
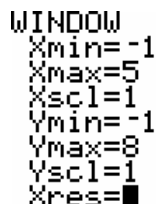
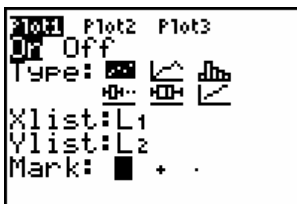
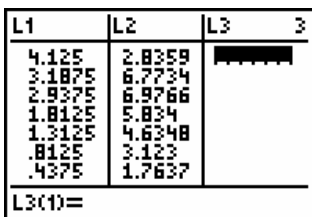
Measuring is very important. Lengths should be measured to the nearest 16^{th} of an inch. Students should complete the table, and then create a scatter plot of the data. The ordered pairs (*height, area*) are the data we want to consider. However, the scatter plot (*height, base*) also has information.

Here is a sample table.

Height (inches)	Base (inches)	Area (inches ²)
4 1/8	1 3/8	2.84
3 3/16	4 1/4	6.77
2 15/16	4 3/4	6.98
1 13/16	6 7/16	5.83
1 5/16	7 1/16	4.63
13/16	7 11/16	3.12
7/16	8 1/16	1.76
3 11/16	3 1/16	5.65

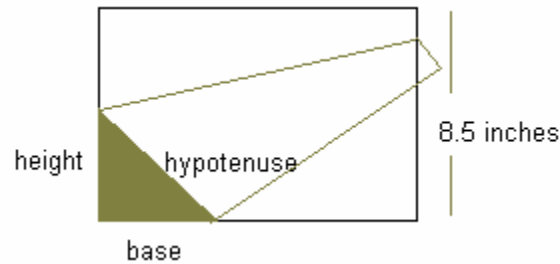
(The structure of the experiment can be more controlled if you specify a group of heights. You could increase by increments of 0.5 inches through 4.)

Put height in List1 and area in List2 to create a scatter plot on the calculator. Screen captures follow.



The scatter plot shows a clear maximum value for area at the point $\left(2\frac{15}{16}, 6.98\right)$ where 6.98 inches² is the area. This is the maximum area for the data set shown above. However, we cannot be sure we have the absolute maximum for all possible triangles. We need to see all the possibilities.

We are in luck since the shape of the scatter plot quickly leads us to believe there is a function that describes this area. If we discover that function, then we can find the maximum of that function. Using the geometry of the problem there are several relationships that can be useful. Consider the diagram.



The sum of the *height* and the *hypotenuse* is 8.5 inches. If $x = \text{height}$, then $8.5 - x = \text{hypotenuse}$. Using the Pythagorean Theorem, we see:

$$(8.5 - x)^2 = x^2 + (\text{base})^2.$$

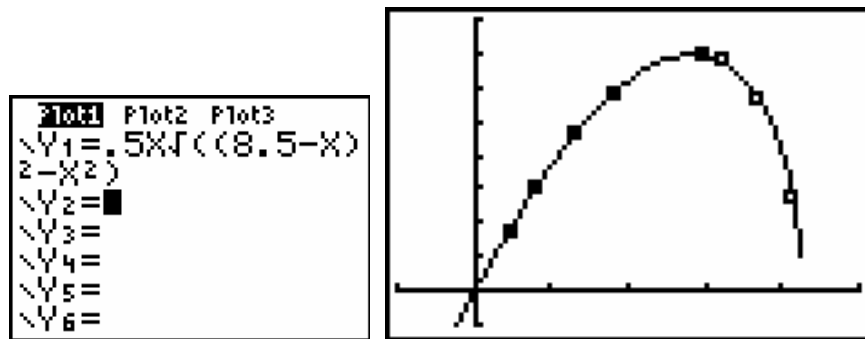
Solve for the *base*

$$\text{base} = \sqrt{(8.5 - x)^2 - x^2}.$$

Therefore, the area is found by

$$\text{Area} = \frac{1}{2} x \sqrt{(8.5 - x)^2 - x^2}.$$

Using this equation in Y1, we can superimpose this curve over the data we collected.

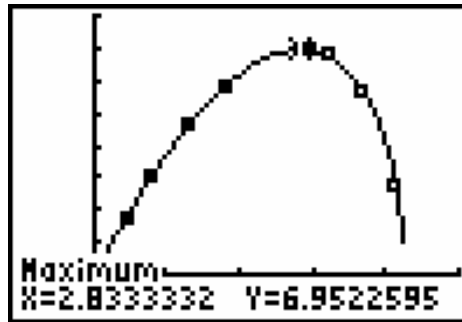


Once we have this function, we can use the 2nd Calc button to find the maximum of the function.

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1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx

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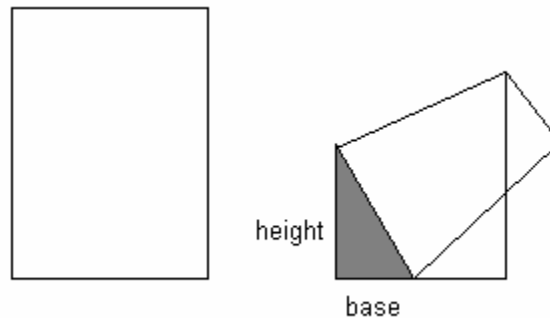
The maximum area from our data set is very close to the maximum of the function. However, (2.83, 6.95) is the approximation of the ordered pair that has a maximum of the area when the paper is in landscape position.

Notice on the graph from the calculator that the right side of the graph seems to disappear or stop. The rate of change is so steep in that region that the technology produces no points to plot there.

However, the shape of the graph on the far right seems to imply that the height values have a limit. Again, based on the geometry of the problem, the height has a maximum value of 4.25 inches which would also produce a value for the hypotenuse of 4.25 inches and a base that was 0. The minimum value of the height is 0 inches. Both the maximum and minimum height-values produce areas of 0 inches². The domain of the function is $0 \leq x \leq 4.25$. Look at the table of values for the area function. Any x -value outside this domain produces negative areas.

Activity Three: Folding Triangles (Portrait)

Once students complete the second activity, activity three is straightforward. In this case the paper is oriented as a portrait with the long side running vertically. This orientation creates different restrictions and constants in the problem, but essentially the form of the function and shape of graph are the same.



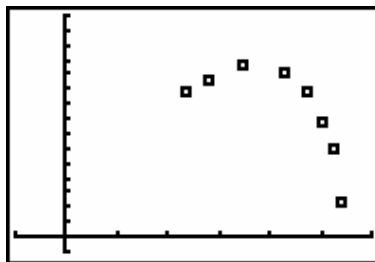
Students can go through the data collection process, but if they are able to write the function to describe area without the data step, that is fine. A sample data list follows with its scatter plot.

Height (inches)	Base (inches)	Area (inches ²)
5 7/16	7/8	2.37
5 1/4	2 5/16	6.07
5 1/16	3 1/16	7.75
4 3/4	4 1/8	9.80
4 5/16	5 1/8	11.05
3 1/2	6 5/8	11.59
2 13/16	7 5/8	10.72
2 3/8	8 5/16	9.87

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WINDOW
Xmin=-1
Xmax=6
Xscl=1
Ymin=-1
Ymax=15
Yscl=1
Xres=█

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The scatter plot indicates that there will be a maximum value for area. From our table, the maximum value of area is approximately 11.59 in².

To find a function, we need to relate lengths using the Pythagorean Theorem. If $height = x$ then $hypotenuse = 11 - x$; the sum of $height$ and $hypotenuse$ is 11 inches. Using the Pythagorean Theorem:

$$(11 - x)^2 = x^2 + (base)^2$$

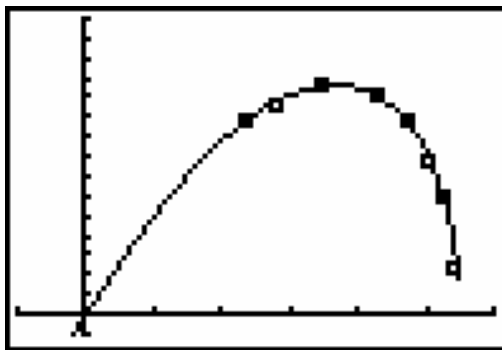
and solving for $base$ we get

$$base = \sqrt{(11 - x)^2 - x^2}.$$

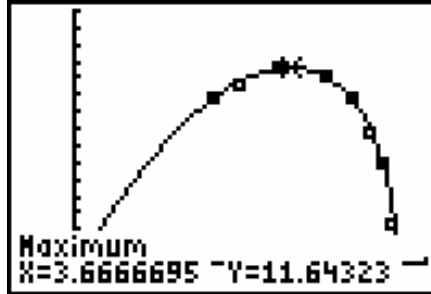
The formula for $area$ will be

$$area = \frac{1}{2}x\sqrt{(11 - x)^2 - x^2}.$$

Put this formula in Y1 and superimpose the graph of that function over the data.



Find the maximum of the function using 2nd Calc feature of the calculator.



The maximum of the function represents all possible areas and is approximately 11.64 in². This result is quite close to the maximum we found in the data—lucky, I guess!

The domain of this function $area = \frac{1}{2}x\sqrt{(11-x)^2 - x^2}$ is $0 \leq x \leq 5.5$.

Activity Four: The Maximum Area

The maximum area with the paper in landscape position is approximately 6.95 inches² when the height is approximately 2.83 inches. The maximum area with the paper in portrait position is approximately 11.64 inches² when the height is approximately 3.66 inches. Therefore, the maximum area from either position is approximately 11.64 inches² when the height is approximately 3.66 inches. Fold that triangle. Compare this result with the results from activity one.

Student Handout

Paper Folding: Maximizing the Area of a Triangle

1. Take a sheet of $8\frac{1}{2}$ by 11 inch paper. Fold the paper so the top left corner of the paper falls somewhere along the bottom edge of the paper. This fold will create a triangle in the left corner of the paper. Write the area and the dimensions of the triangle of largest area. Compare triangles with others made by members of your group.

As you can see from the folding there are many options. To really determine the triangle of largest area, we need some mathematical help.

2. You will need a ruler, a sheet of $8\frac{1}{2}$ by 11 inch paper, and your calculator. For this investigation, hold the paper in landscape position— $8\frac{1}{2}$ inches as the vertical and 11 inches along the horizontal.
 - a. Fold 8 different triangles. For each, measure the height and base of the triangle and calculate the area. Fill in the table below.

Height								
Base								
Area								

- b. Create a scatter plot of the ordered pairs (*height*, *area*).
 - c. Based on the scatter plot, determine the triangle with maximum area.
 - d. Using geometry create a function that describes the area of the triangle given the height. Superimpose the graph of this function over the scatter plot of the data.
 - e. Find the maximum area based on the values of the function.
 - f. Compare the maximum values from c and e above. Determine and support which is best.
 - g. What values of *height* are feasible? What is the domain of the function you found in d above?
3. Complete a similar investigation, but this time put the paper in portrait position— $8\frac{1}{2}$ inches as the horizontal and 11 inches as the vertical. Determine the maximum area for this triangle.
 4. What is the absolute maximum area that can be created by folding the paper so the top left corner of the paper falls somewhere along the bottom edge of the paper?