

Exponential Functions: Half Life and Doubling Time Algebra 2

Goals:

1. Write and graph exponential functions of the form $f(x) = ab^x$ (3.15)
2. Use exponential functions to solve problems. Solve by: graphing, substitution, and using properties of equality (justify steps) (3.17)
3. Write and interpret an equation of a curve (exponential) which models a set of data. (4.01)
4. Find the equation of curve of best-fit (exponential) for a set of data. Interpret the constants, coefficients, and bases in the context of the data. Check the equation for goodness of fit and use the equation for predictions (4.02).
5. Use exponential equations of the form $f(x) = (1 + r)^x$ where r is given as a rate of growth or decay to solve problems.

Equipment and materials needed for students:

1. Copy of handout
2. Graphing calculator
3. Paper and pencil for note taking.
4. If students do the m&m's experiment in groups, each group will need a paper plate, at least 50 m&m's, and a small cup for tossing the candies on the paper plate.

Activities One: Half Life

Using m&m's (can use Skittles or pennies), count the number of pieces of candy (minimum 50 pieces). Record this count in the table below by the throw number of 0. Place the candies in a cup or bag, shake, and throw them onto a flat contained space (like a paper plate). Remove all m&m's that show the "m" (if pennies are used, select either heads or tails to remove). Count the number remaining and record in the table below by throw number 1. Continue the process always recording throw number and number of m&m's until one remains.

throw #	0	1	2	3	4	5	6
pieces							

1. Make a scatter plot of the ordered pairs (throw #, number of pieces). Discuss the general shape of the data.
2. Find the model of best fit for the scatter plot. Check the residuals to verify you have a good fit.
3. In general, how many pieces do you pull out with each throw?

The purpose of this example is to illustrate the process of the number being cut in half with each throw. An example of a data set is:

throw #	0	1	2	3	4	5	6
pieces	49	24	10	6	2	2	

The model achieved should be nearly $y = k\left(\frac{1}{2}\right)^t$, where k is the number of pieces in the beginning, t is the throw number, and y is the number of pieces remaining after that throw.

This example illustrates the idea of half-life. In a **decreasing** (or **decaying**) exponential function, the **half-life** is the interval of time over which the amount is halved. This is generally used with chemical substances like radiation.

As an example, thorium-234 has a half-life of 25 days or carbon 14 has a half-life of 5730 years. Show students by plotting points what this means. May show the equation

$$y = k\left(\frac{1}{2}\right)^{\frac{t}{\text{half-life}}}$$

if students are interested.

Activity Two: Doubling Time

A sheet of paper is cut in half. The resulting pieces are stacked on top of each other and cut in half again and stacked again. Continue this procedure at least seven times. Complete the table below to show at the end of the cut the number of pieces of paper.

cut #	0	1	2	3	4	5	6	7
sheets	1							

1. Make a scatter plot of the ordered pairs (cut #, number of sheets). Discuss the general shape of the data.
2. Find the model of best fit for the scatter plot. Check the residuals to verify you have a good fit.
3. In general, what happens to the number of sheets of paper with each cut?

The sheets of paper should double with every cut.

cut #	0	1	2	3	4	5	6	7
sheets	1	2	4	8	16	32	64	128

You will probably not be able to make the last cut. The stack is too thick.

The model that will fit this data is $y = 1(2)^t$. The value of 1 shows the original amount and t illustrates the cut number.

The purpose of this example is to illustrate doubling time. In this case every time a cut is made, the number of sheets of paper doubles. In an **increasing** exponential function, the **doubling time** is the interval time over which a substance doubles. This concept is often used to describe population growth.

For example, in the twentieth century, the world population doubled every 51.8 years. The world population at the beginning to the twentieth century was 1.6 billion. At the end of the century it was 6.1 billion. An equation that describes this growth is

$$y = 1.6 \cdot (2)^{\frac{t}{51.8}}, \text{ if students are interested.}$$

Activity Three: Using this idea.

Remember the Fran Data:

Hurricane Fran hit North Carolina on the evening of September 5, 1996. Over one million homes and businesses were left without power. Repair crews began immediately restoring electrical service. This data is taken from the Algebra II Indicators prepared by DPI.

Date	Customers without power
Sept 6	1,159,000
Sept. 7	804,000
Sept. 8	515,000
Sept. 9	340,500
Sept. 10	195,200
Sept. 11	136,300
Sept. 12	77,000
Sept. 13	37,600

1. Based on the graph of the data, which equation (linear, exponential, or quadratic) is most likely to generate a good algebraic model of the hurricane recovery?
2. According to the model, describe the recovery effort.
3. If the recovery continued at the same pace, how long until all power was restored (less than 1,000 customers without power)?
4. Is it appropriate to discuss half-life or doubling time with this data? Explain. When decided find the value of the appropriate one.

Now that we know about the exponential function, we can see that this data appears to be exponential. It seems to be decaying and by observation, we can guess a half-life—maybe 2 days.

Have students create a scatter plot using the ordered pairs (day #, customers). Begin with Sept. 6 as day 0.

Fit the data using an exponential model: $y = 1306850 \cdot (0.6191)^x$. A review of the residuals shows this to be a good model.

Describe the recovery effort: Each day fewer customers were without electricity; approximately 62% of the number of customers without electricity remained without electricity the next day. In this question, test the use of the ANS button on the calculator. Begin with 1,159,000 and generate values to go with $0.6191 \cdot \text{ANS}$ to compare to the data. Some values are high while others are low. Discuss issues of curve fitting.

When will fewer than 1000 customers be without electricity. Graph the line $y = 1000$ on the graph and find the point of intersection. Beware; you will need some change in the

window. This occurs at the point (14.97, 1000) that means that 15 days after Sept. 6 has everyone fixed.

Because this exponential is decreasing, there must be a half-life. To determine the half-life, take several y – values that illustrate halving and find the x – values. For example, let $y = 1,000,000$ then let $y = 500,000$, then $y = 250,000$ and find their x – values using intersect. The following ordered pairs are found: (0.558,1000000), (2.004,500000), and (3.45,250000). By finding the differences between the x – values, we find the half-life is approximately 1.5 days.

Follow-Up Problems

- Xerox machines have the ability to reduce the size of an image onto a photocopy. Often the options of size reduction are limited. If a xerox machine will only reduce an image to 90% of the original size, you could produce the 90% reduced image and then xerox that image to produce an image that was 81% of the original ($0.9 \cdot 0.9$). How many times would you have to reduce the image in order to produce an image that was 50% of the original size? What would be the half-life of this reduction procedure?

The function that models the reduction is $R(x) = k(0.9)^x$, where k = original dimension, x = number of reductions, and R = resulting size. Assume the original is a size of 1 inch, put the function $R(x) = (0.9)^x$ in Y1 of the calculator and use the table of values to determine when R is 0.5. Also, you could graph the function and the line $y = 0.5$ to determine where they intersect.

The result is that the image is about half size on the 6th or 7th reduction. Interesting to add that decimal values are not appropriate as answers—must be whole number.

- The North Carolina population is counted with every census. In the table below, the populations are given for every other census. What is the doubling time for the population of North Carolina?

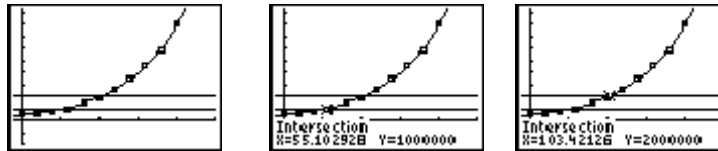
year	population (to nearest 1000)
1800	478,000
1820	639,000
1840	753,000
1860	993,000
1880	1,400,000
1900	1,894,000
1920	2,559,000
1940	3,572,000
1960	4,556,000
1980	5,880,000
2000	8,049,000

This problem is taken from the Algebra II Indicators developed by the NC Department of Public Instruction.

Move the data so that the year 1800 is represented by $x = 0$. Using exponential regression fit, we produce the equation $P(x) = 453629 \cdot (1.0144)^x$ where $x =$ years since 1800 and $P =$ population of NC. This produces a very nice fit.

For students to determine the doubling time, this could be done algebraically by solving the equation $453629(1.0144)^x = 907258$.

Using the graph, a student can pick a population-value within the existing range and another population-value that is twice the first one selected. Show the function and the two horizontal lines on the same graph. Using intersect option, find the points of intersection. The doubling time will be the difference between the two year-values. In the example, the lines used were $y = 1000000$ and $y = 2000000$.



This tells us that the doubling time is $103.4 - 55.1 = 48.3$ years.

Student Handout
 Half-life and Doubling Time
 Algebra 2

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Follow-Up Problem

Half-life and Doubling Time with Exponential Functions

Algebra 2

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