

Polynomials as Models

Algebra 2

Goals:

1. Describe graphically, algebraically and verbally real-world phenomena as functions; identify the independent and dependent variables. (3.01)
2. Use polynomial equations to solve problems. Solve by graphing. (3.07)
3. Find zeros, intercepts, and approximate turning points of polynomial functions; describe them in the context of the problem. (3.08)
4. Use systems of two or more equations to solve problems. Solve by using matrix equations of the form $AX=B$. (3.12)
5. Operate with matrices to solve problems. Find the inverse of a matrix. (4.04)

Materials and equipment needed for each student:

1. Copy of the student handout
2. Graphing calculator
3. Paper and pencil for note taking

Activity One: Look before you leap. Be sure to read the question.

This question is taken from the Algebra 2 Indicators: problem 3.12 B. This problem was developed by the NC State Department of Education as an example of a kind of question for the End of Course Test.

$(1,7)$, $(6,-2)$, $(11,3)$, and $(15,-6)$ are points on the graph of $y = ax^3 + bx^2 + cx + d$. Using matrix equations of the form $AX = B$, set up the matrix equation to determine a , b , c , and d . What is the equation? What is A^{-1} ?

This question implies that if we have four points we can find a cubic function that passes through these points. After the examples done above, data analysis looks like a good option. Put the x -values of the points in L1 and the y -values of the points in L2.

Use the cubic regression line option of the Stat Calc menu. This gives the polynomial:



However, this is not what the question asks. We need to set up a matrix equation. To do this we must substitute each ordered pair in the equation to create a new equation that connects the variables a , b , c , and d . The four resulting equations are:

$$7 = a + b + c + d \text{ for the point } (1,7)$$

$$-2 = 216a + 36b + 6c + d \text{ for the point } (6,-2)$$

$3 = 1331a + 121b + 11c + d$ for the point (11,3)
 $-6 = 3375a + 225b + 15c + d$ for the point (15, -6).

This gives the matrix equation:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 216 & 36 & 6 & 1 \\ 1331 & 121 & 11 & 1 \\ 3375 & 225 & 15 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 3 \\ -6 \end{bmatrix}$$

which answers the first question.

The value of A^{-1} is found by using the calculator:

$$\begin{bmatrix} -0.001 & 0.004 & -0.005 & 0.002 \\ 0.046 & -0.12 & 0.11 & -0.036 \\ -0.459 & 0.849 & -0.555 & 0.165 \\ 1.414 & -0.733 & 0.45 & -0.131 \end{bmatrix}$$

The more important question is do we get the same cubic polynomial that cubic

regression found? To determine this, we must actually perform the product of $A^{-1} \cdot \begin{bmatrix} 7 \\ -2 \\ 3 \\ -6 \end{bmatrix}$.

The result is $\begin{bmatrix} -0.04579 \\ 1.10429 \\ -7.56087 \\ 13.50238 \end{bmatrix}$ which show the same coefficients that were found using the

curve fitting techniques on the calculator.

Activity Two: Gas Prices as a polynomial

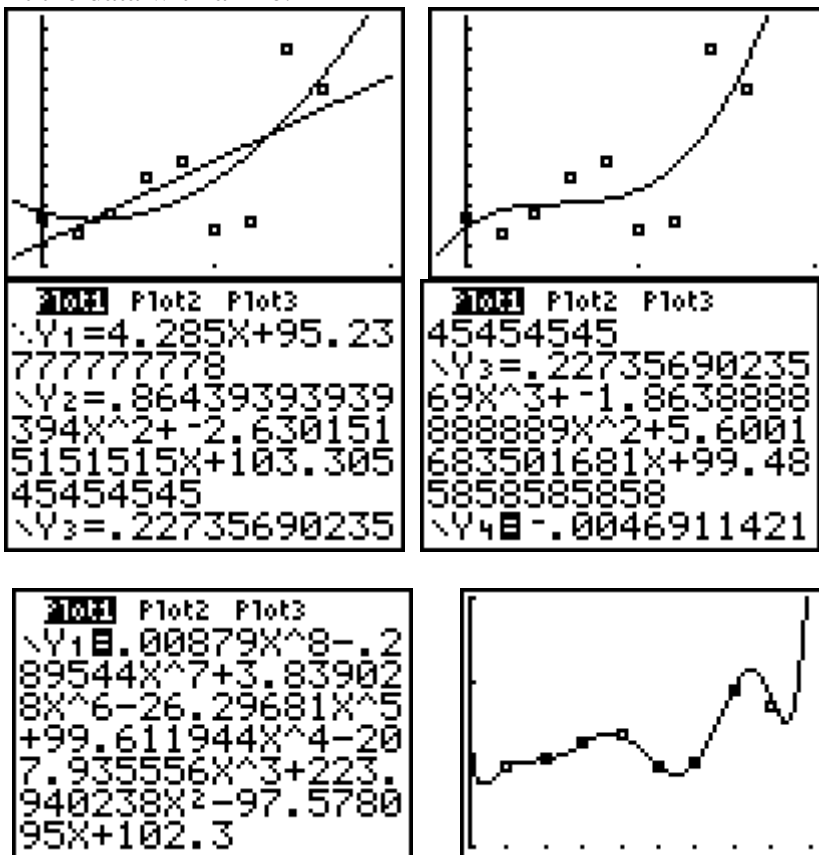
The data shown below represents the average cost of gasoline per gallon in the lower Atlantic states on the first of April for years 1993 to 2001. The data is converted to change the year to the number of years since 1993. The cost is given in cents.

yr	0	1	2	3	4	5	6	7	8
cost	102.3	97.6	102.6	112.3	116.7	98.6	101	145.2	135.1

- Create a scatter plot of the data (year, cost).
- Find the function that will best fit this data. Defend your choice using characteristics of the functions and use of residuals.

- c) Find a linear function to fit the data. Find a quadratic, cubic, and quartic functions to fit the data. Which seems to be the best model?
- d) Use each function to estimate the price per gallon of gas for 2002.
- e) Polynomials move all around. Try this polynomial over the data:
- $$y = 0.00879x^8 - 0.28954x^7 + 3.839028x^6 - 26.29681x^5 + 99.611944x^4 - 207.935556x^3 + 223.940238x^2 - 97.578095x + 102.3$$
- f) Find the function that will best fit this data. Defend your choice using characteristics of the functions, general trend and use of residuals.
- g) Predict the average price per gallon of gasoline in the lower Atlantic states in April of 2002.

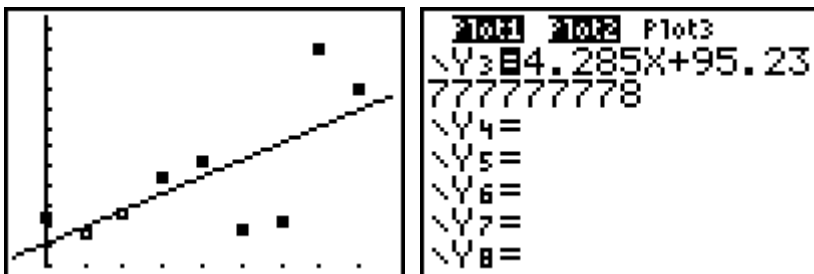
Some possible answers for the questions above are shown in the graphs below which were taken from the calculator. Perhaps the best answer to this particular problem is to fit the data with a line.



This polynomial was found using matrices. Depending on time constraints and readiness, this may be a topic that you will want to discuss with students.

The best model appears to be the line. If this model is used, the average price per gallon of gasoline in 2002 (found by substituting $x = 9$ into the function) is 133.8 cents.

The other estimates for 2002 were \$149.65 from the quadratic model, \$164.66 from the cubic model, and \$163.51 from the quartic model. The 8th degree polynomial estimates \$268.58 per gallon in 2002.



The data for this problem can be found on the website:

http://www.eia.doe.gov/oil_gas/petroleum/data_publications/wrgp/mogas_home_page.html

Follow Up Activity: Summation formulas and the polynomial

In this lesson, students will gather data in small groups based on the concept of summation. Each data set will then be fit with either a quadratic, cubic, or quartic equation.

a. Sum of the first k integers:

number k	1	2	3	4	5	6
sum of the first k integers						

b. Sum of the first k odd numbers:

number k	1	2	3	4	5	6
sum of the first k odd integers						

c. Sum of the first k integers squared:

number k	1	2	3	4	5	6
sum of the first k integers squared						

Each of these data sets will be fit perfectly by a polynomial function. The formulas that you may remember from Calculus are listed beside the polynomials that students will find with data analysis.

The following polynomials result:

a. Sum of the first k integers or $\sum_{k=1}^n k$: $y = 0.5x^2 + 0.5x$ or $\sum_{k=1}^n k = \frac{n(n+1)}{2}$

b. Sum of the first k odd numbers or $\sum_{k=1}^n (2k-1)$: $y = x^2$ or $\sum_{k=1}^n (2k-1) = n^2$

c. Sum of the first k integers squared or $\sum_{k=1}^n k^2$: $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x$ or

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

This activity can include the entire class doing one of these or assigning different activities to different groups. The teacher may show students the sigma notation used in finding these sums. Since coefficients given by the calculator are in decimal form, students should convert decimal values to the appropriate fraction to see the resulting formulas. You may want to share the summation notation formulas with them.

Note: these formulas work perfectly because there is no error in the data.

Student Handout
Polynomials
Algebra 2

1. $(1,7)$, $(6,-2)$, $(11,3)$, and $(15,-6)$ are points on the graph of $y = ax^3 + bx^2 + cx + d$. Using matrix equations of the form $AX = B$, set up the matrix equation to determine a , b , c , and d . What is the equation? What is A^{-1} ?

This question is an Algebra 2 Indicator created by the NC Department of Public Instruction.

2. The data shown below represents the average cost of gasoline per gallon in the lower Atlantic states on the first of April for years 1993 to 2001. The data is converted to change the year to the number of years since 1993. The cost is given in cents.

yr	0	1	2	3	4	5	6	7	8
cost	102.3	97.6	102.6	112.3	116.7	98.6	101	145.2	135.1

- a) Create a scatter plot of the data (year, cost).
- b) Find a linear function to fit the data. Find a quadratic, cubic, and quartic function to fit the data. Compare the residuals of each model.
- c) Use each function to estimate the price per gallon of gas for 2002.
- d) Polynomials move all around. Try this polynomial over the data:

$$y = 0.00879x^8 - 0.28954x^7 + 3.839028x^6 - 26.29681x^5 + 99.611944x^4 - 207.935556x^3 + 223.940238x^2 - 97.578095x + 102.3$$
 What does this model predict for the price per gallon of gasoline for April 2002?
- e) Find the function that will best fit this data. Defend your choice using characteristics of the functions, general trend and use of residuals.

Follow Up Problem
Polynomials
Algebra 2

We will use only the positive integers in these questions.

- First fill in the table
- Using the value of k as your independent variable and the sum as the dependent variable, make a scatterplot of the data
- Fit the data with an appropriate polynomial function
- Use your function to find the value of the sum if $k=15$.

1. Find the sum of the first k integers:

If k is	1	2	3	4	5	6
Find sum of the first k integers						

2. Find the sum of the first k odd integers:

If k is	1	2	3	4	5	6
Find sum of the first k odd integers						

3. Find the sum of the first k integers squared

If k is	1	2	3	4	5	6
Find sum of the first k integers squared						