This table shows exits and tolls on the New Jersey Turnpike. When a car enters the Turnpike, the driver is given a ticket specifying the entry exit. When the driver exits the Turnpike, the total toll is collected.

1. Use a map to become familiar with the Turnpike and its exits.
2. Using mileage and cost create a scatter plot of the data.
3. The data point at exit 6 seems out of place on the scatter plot. If you take exit 6, where will it take you? Remove exit 6 from your data set and re-create the scatter plot.
4. Which function best fits the data: a linear, a quadratic, or an exponential function? Support your answer. Use residuals to help with this judgment.
5. The data seems to be in two parts—both linear. Separate the data and find models for each part. Write a piecewise defined function for the data and discuss goodness of fit.
6. For each linear expression developed to fit the data, explain the meaning of the slope and the y-intercept in the context of the problem.
7. What toll would your model predict for: 50 miles (Exit 6), 95 miles (Exit 12) and 112 miles (Exit 16W)?
8. If a new exit is added between Exits 2 and 3 at 20 miles, what would be an appropriate cost?
9. Go back to a map or description to look at the two parts of the data. Is there a geographical reason for the tolls having two models? If so, what is the reason? Which of the two models is most expensive? Why?
10. If we had studied the trip on the toll road beginning at Exit 16W and traveling to Exit 1 (from North to South), how would the models be similar and how would they be different?
Tolls on the New Jersey Turnpike
Advanced Functions and Modeling

Students develop a model for the toll structure of the New Jersey turnpike using a piecewise defined linear function developed using data analysis. This problem is based on work done by John A. Goebel for his Algebra 2 class at Durham Academy in 2003.

Goals Addressed in the Lesson:
1. Create and use calculator-generated models of a linear function of bivariate data to solve problems. (1.01)
   a. Interpret the constants and coefficients in the context of the data.
   b. Check models for goodness of fit; use the most appropriate model to draw conclusions and make predictions.
2. Use piecewise-defined functions to model and solve problems; justify results. (2.02)
   a. Solve using tables, graphs, and algebraic properties.
   b. Interpret the constants and coefficients in the context of the problem.

Materials Needed:
1. Copy of the student handout for each student.
2. Graphing calculator for each student.
3. Graph paper.
4. Access to the internet (if possible).
5. Map of New Jersey or web link to New Jersey map.

Activity One:
The goal of this activity is to develop interest in the issue of toll roads—how do they work, what do they look like, why they exist? We want the students to know something about the New Jersey Turnpike since the data in this problem is from there. Several particular exits will be of interest so students need a way to determine where an exit takes them. Some avenues to help students get acquainted are listed below.

- Look at maps of New Jersey
- Go to [http://www.state.nj.us/turnpike/nj-vcenter-maps.htm](http://www.state.nj.us/turnpike/nj-vcenter-maps.htm) to view maps of the New Jersey Turnpike.
- A list of the interchanges and their locations can be found at [http://www-contrib.andrew.cmu.edu/~mn2n/tollroads.html](http://www-contrib.andrew.cmu.edu/~mn2n/tollroads.html).
- A website with a number of cameras on different parts of the New Jersey Turnpike can be seen at [http://newyork.metrocommute.com/video/newyork/indexNJTpke.html](http://newyork.metrocommute.com/video/newyork/indexNJTpke.html).

Activity Two:
The following data show the exit number, mileage (from first exit) and toll for the New Jersey Turnpike. These data are for peak hours using an E-Z Pass. The E-Z Pass is an
1. Create a scatter plot of the data showing cost versus mileage.

![Scatter plot of cost versus mileage]

Notice the point associated with exit 6 does not seem to fit the trend around it. Delete that point and reconsider the data. Exit 6 takes a driver to the Pennsylvania Turnpike.

4. Fit this data with a linear, quadratic, and exponential model. Discuss goodness of fit for each model. Which model is best?

*Note on Goodness of Fit:* We will explore goodness of fit through residuals. A residual is defined as the vertical distance between a data point and the model describing the data. The value of a residual is calculated using the following formula

For a given $x_{data}$, the residual $= y_{data} - y_{model}$. 

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The values of the residuals are found on the calculator using the lists. Given the $x$-values of the data in list L1, the $y$-values of the data in L2 and the model for the data in Y1, go to the top of L3 and type $L3 = L2 - Y1(L1)$.

<table>
<thead>
<tr>
<th>L1</th>
<th>L2</th>
<th>L3</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.56</td>
<td>-----</td>
</tr>
<tr>
<td>20</td>
<td>1.15</td>
<td>-----</td>
</tr>
<tr>
<td>30</td>
<td>1.45</td>
<td>-----</td>
</tr>
<tr>
<td>40</td>
<td>1.75</td>
<td>-----</td>
</tr>
<tr>
<td>50</td>
<td>2.05</td>
<td>-----</td>
</tr>
</tbody>
</table>

A residual plot is created by plotting the ordered pair $(x_{\text{data}}, \text{residual})$. The residual plot should only show noise and should have no pattern. Otherwise, the model used may not be the best model for that data.

**Linear Model:** $y = 0.0473x - 0.666$ where $x = \text{mileage}$ and $y = \text{cost of toll}$.

On the left, the linear model is shown below superimposed over the data. The residuals verify that this model is not a satisfactory one. The residuals are shown in the middle graph and its window is on the far right.

**Quadratic Model:** $y = 0.0049x^2 - 0.02x + 1.07$ where $x = \text{mileage}$ and $y = \text{cost of toll}$.

On the left, the quadratic model is shown below superimposed over the data. The residuals verify that this model is again not satisfactory (seems a bit better than the line). The residuals have a definite pattern which verifies the concern with the model.

**Exponential Model:** $y = 0.5243 \cdot (1.0205)^x$ where $x = \text{mileage}$ and $y = \text{cost of toll}$.

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On the left, the exponential model appears to fit the data; however, the residuals (shown in the middle) show a pattern. Thus, the exponential is also not a good model for this data. This fit is very similar to that of the quadratic function.

As we look at these three possible models, the exponential has residuals that are a bit smaller than those of the quadratic function, but a clear pattern is still visible in every residual plot. We need to consider other models.

5. Separate the data into two parts. This will depend on how the student sees the data. For the purpose of producing a solution, we will break the data at exit 12 which is at 95 miles. The point associated with exit 12 will be included in both data sets for the purpose of fitting the line. When we write the model, the point at 95 miles must go with only one part of the piecewise function.

Students may select different breaking points and not choose to include the point in both sets. Hence, there are many answers. This is one example.

For exits 1 through 12 or $0 \leq x \leq 95$, the regression line (using LinReg from the Stat Calc menu) is $y = 0.0302x + 0.1249$ where $x$ = the number of miles from exit 1 and $y$ = the cost of the toll when you exit. The line is shown superimposed over the data followed by the residual plot associated with this linear fit.
The residual plot shows that the line fits the beginning points best. The residuals show a fairly random shape as well as the value of the residuals is small which is shown by the $y$-values of the window.

The second part of the data for $95 \leq x \leq 112$ or exits 12 through 16W is fit by the regression line $y = 0.1117x - 7.2652$ where $x =$ the number of miles from exit 1 and $y =$ the cost of the toll when you exit. The line is shown superimposed over the data followed by the residual plot associated with this linear fit.

This residual plot shows a very scattered set of points and reflects that the line is a good model for this data.

The middle screen below shows the creation of a piecewise function on the calculator. Find Y1 and Y2 on the VARS menu and the inequalities under TEST. A scatter plot of the full data set with the two lines superimposed over the data is shown.
The slopes and y-intercepts have meaning to the NJ Turnpike.

- For the function \( y = 0.0302x + 0.1249 \) where \( x \) = the number of miles from exit 1 and \( y \) = the cost of the toll when you exit. This function represents exits 1 through 12 or \( 0 \leq x \leq 95 \). The slope of 0.0302 measures the cost in dollars per mile on Turnpike for these exits. More generally, the slope represents a cost of 3 cents per mile. The y-intercept 0.1249 measures the initial cost in dollars for just entering the Turnpike.

- For the function \( y = 0.1117x - 7.2652 \) which represents \( 95 \leq x \leq 112 \) or exits 12 through 16W, the slope 0.1117 measures cost in dollars per mile. More simply, it costs 11 cents per mile to travel between exits 12 and 16W. The y-intercept is not really meaningful since \( x \) cannot be 0.

Use the function \( t(x) = \begin{cases} 0.0302x + 0.1249, & 0 \leq x \leq 95 \\ 0.1117x - 7.2652, & 95 < x \leq 112 \end{cases} \) to find values of tolls.

- At \( x = 50 \), the model produces a toll of $1.64. At Exit 6, the toll is $2.20.
- At \( x = 95 \), the model predicts a toll of $2.99. At Exit 12, the toll is $3.20.
- At \( x = 112 \), the model predicts a toll of $5.24. At Exit 16W, the toll is $5.00.
- If a new exit is added at \( x = 20 \), the model predicts a toll of $0.72. For ease it would be changed to $0.75.

9. A review of the map shows that once we pass Exit 12, the NJ Turnpike travels through a dense population area near New York City. The cost per mile greatly increases but also the cost of the turnpike does as well. There will have to be more lanes and more repairs and changes in the turnpike since it serves some many more people. Clearly, the most expensive travel is from exits 12 through 16W.

**Possible Extension**

Once this problem has been explored, students could develop a toll system for I-95 in North Carolina. To complete this task, students would need maps with I-95 exits and distances shown. There have been news articles with information about major goals. For example, follow the link to [http://www.newsobserver.com/front/digest/story/3025530p-2771057c.html](http://www.newsobserver.com/front/digest/story/3025530p-2771057c.html). Perhaps as students produce their toll system, they can work to have each car that travels the full length of I-95 in North Carolina pay a total of $18, while establishing toll booths as they exit I-95 at certain exits.