I. Introductory material
Parametric equations can be used to model many real-world situations. The following example opens the door to the topic. We will revisit this example later as we develop our skills in working with parametric equations.

Ring Toss Problem
Each year Sam looks forward to the ring toss at the county fair. Sam is not happy to hit just any peg, but instead he prides himself on hitting the middle peg in the front row each year. Suppose Sam tosses the ring with an initial horizontal velocity of 8 feet/sec and an initial vertical velocity of 3 feet/sec and that he’ll release the ring when the ring is 4 feet above the ground. Sam lines up directly in front of the middle ring at a distance of 4 feet from the ring platform. Our job is to model the motion of the ring using parametric equations and find out whether or not Sam hits his target – the middle peg. The Ring Toss game is shown below. It sits on a short stool measuring 1 foot in height. The pegs are two inches high and the bottom of the first row of pegs is 4 inches from the bottom of the game.

We will work through more basic problems before we tackle this one.

II. Background discussion of mathematics
In order to work with the parametric equations in this section, the students should be familiar with using linear, polynomial, and trigonometric functions as models. The textbooks Functions Modeling Change and Algebra and Trigonometry have sections explaining the concepts of how parametric equations are used as models, and they each have problems in which the students can experiment with the graphs of parametric equations. In addition to these textbooks, Contemporary Precalculus Through Applications, NCSSM, Barrett et al (Glencoe) also has a section on parametric equations.

III. Ties to sections in textbooks
In Functions Modeling Change, see Section 12.1 and in Algebra and Trigonometry, see section 11.8.
IV. **Worked example** based on background mathematics

The following examples give you some ideas on how to introduce the topic.

The Ring Toss Problem that follows contains teacher notes which provide specifics on how to use parametric mode on the calculator and how to produce numeric values and graphs in parametric mode.

**Example 1:** The motion of a projectile at time $t$ (in seconds) is given by the parametric equations:

$$x(t) = 25t$$
$$y(t) = -16t^2 + 30t + 10$$

Where $x(t)$ gives the horizontal position of the projectile in feet and $y(t)$ gives the vertical position of the projectile in feet.

**a.** Find the vertical and horizontal position of the projectile when $t = 2$.

**Solution:** Graph the equations on the calculator in Parametric Mode. When you type the equations in your Y=, you should see the following. Notice when you hit the X,T, θ key on the calculator, you get a T instead of an X, θ.

Using the Window Tmin = 0, Tmax = 5, Tstep = 0.2, Xmin = -1, Xmax = 70, Xscl = 10, Ymin = 0, Ymax = 50, Yscl = 10, we see

Using TRACE, we can find that $x = 50$ and $y = 6$ when $t = 2$. The units for the $x$ and $y$ coordinates are feet.

**b.** At what time will the projectile hit the ground?

**Solution**

If we choose to use our calculator, we cannot use the ZERO function under the CALC menu, so we should make the Tstep smaller and use TRACE to find the $t$ value when
\( y = 0 \). With a \( T \text{step} = 0.01 \), the projectile hits the ground between 2.16 and 2.17 seconds. We can choose to solve this analytically by setting \( y(t) = -16t^2 + 30t + 10 = 0 \) and solving for \( t \) using the quadratic formula.

**Example 2:**
Taken from *Precalculus Through Applications, NCSSM, Barrett et al.*

The parametric equations below represent the hawk and dove populations at time \( t \), where \( t \) is measured in years.

\[
\begin{align*}
  h(t) &= -10 \cos \left( \frac{\pi t}{2} \right) + 20 \\
  d(t) &= 100 \sin \left( \frac{\pi t}{2} \right) + 150
\end{align*}
\]

**a.** Use your calculator in function mode to graph the hawk and dove populations over time.

**Solution:** Make sure your calculator is in Radian mode for this problem. In function mode your Y= should look like:

```
Plot1 Plot2 Plot3
Y1=-10cos(\pi X/2)
+20
Y2=100sin(\pi X/2)
+150
Y3=
Y4=
Y5=
```

The graph with the WINDOW Xmin = 0, Xmax = 15, Xscl = 5, Ymin = 0, Ymax = 300, Yscl = 50 looks like

```
```

**b.** Find the maximum and minimum values for each population.

**Solution:**
Minimum value for the hawk population is 20 and the maximum value is 30.
Minimum value for the dove population is 50 and the maximum value is 250.

**c.** Now using Parametric mode on your calculator, graph the hawk population versus the dove population.
Solution:

\begin{align*}
\text{Plot1} & \text{ Plot2 Plot3} \\
X1T & = -10\cos(\pi T/2) + 20 \\
Y1T & = 100\sin(\pi T/2) + 150 \\
X2T & = \\
Y2T & = \\
X3T & =
\end{align*}

Using a WINDOW as follows: Tmin = 0, Tmax = 5, Tstep = 0.1, Xmin = 0, Xmax = 35, Xscl = 5, Ymin = 0, Ymax = 300, Yscl = 50, we should see

\[X1T = -10\cos\frac{\pi T}{2}, \quad Y1T = 100\sin\frac{\pi T}{2}\]

\[T=0, \quad X=10, \quad Y=150\]

**d.** Using the parametric graph, find the population of hawks and doves after one year.

**Solution:**
By using TRACE, we get 20 hawks and 250 doves.

**e.** When will the population of hawks reach its maximum value and what is that value? How does this value compare with your answer in part **b**?

**Solution:**
Using trace, we see that the maximum hawk population is reached after 2 years and that maximum value is 30. This value agrees with the value we found in part **b**.

**V. Suggestions for homework**

**Functions Modeling Change** Section 12.1

**Algebra and Trigonometry** Section 11.8
Pages 804 – 805 Problems 1 – 26 Problems 35 – 42.

**VI. Suggestions for projects**

There are two projects or activities supplied here. The first activity walks the students through writing the parametric equations with the Ring Toss problem that introduced the section.
The second requires the students to write linear parametric equations for the path of the Mars rover, Spirit. The solutions are provided as well as student handouts for each problem.

**Introduction to Parametric Equations – Ring Toss Problem: Teacher Notes**

**Advanced Functions and Modeling**

**Goal: Optional Functions for Additional or Continued Study**

**Equipment and materials needed for students:**
1. Copy of handout, 1 Student Handout for each student.
2. Graphing calculator
3. Paper and pencil for note taking.

This problem is designed to introduce parametric equations. The students will use the calculator to generate numerical values and graph parametric equations to model the motion of a ring in a ring toss game.

Each year Sam looks forward to the ring toss at the county fair. Sam is not happy to hit just any peg, but instead he prides himself on hitting the middle peg in the front row each year. Suppose Sam tosses the ring with an initial horizontal velocity of 8 feet/sec and an initial vertical velocity of 3 feet/sec and that he’ll release the ring when the ring is 4 feet above the ground. Sam lines up directly in front of the middle ring at a distance of 4 feet from the ring platform. Our job is to model the motion of the ring using parametric equations and find out whether or not Sam hits his target – the middle peg. The Ring Toss game is shown below. It sits on a short stool measuring 1 foot in height. The pegs are two inches high and the bottom of the first row of pegs is 4 inches from the bottom of the game.

![Ring Toss Game](image)

We will think about the horizontal and vertical positions of the ring separately and write functions for each. These functions will be written as functions of time. Let \( x(t) \) represent the horizontal position of the ring at time \( t \) and \( y(t) \) represent the vertical position of the ring at time \( t \), where \( t \) is measured in seconds and the positions are
measured in feet. These equations are called parametric equations and \( t \) is called the parameter.

Consider the horizontal position first. Since we know that Sam tosses the ring with a horizontal velocity of 8 ft/sec and that distance = rate x time, our function is \( x(t) = 8t \).

Now for the vertical position, we know that the vertical position of the ring will be given by \( y(t) = -\frac{1}{2}gt^2 + v_0t + y_0 \), where \( g \) is the acceleration due to gravity (32 ft/sec\(^2\)), \( v_0 \) is the initial velocity and \( y_0 \) is the initial height of the ring when Sam releases it. So we have, \( y(t) = -16t^2 + 3t + 4 \).

Using these equations, we can use our calculator in parametric mode to model the motion of the ring.

On the calculator hit the Mode Key, arrow down to where you have Func Par Pol Seq Highlight Par and hit Enter. Now hit the \( \text{Y=} \) key. We see the following:

```
Plot1 Plot2 Plot3
\( x_1t = \)
\( y_1t = \)
\( x_2t = \)
\( y_2t = \)
\( x_3t = \)
\( y_3t = \)
\( x_4t = \)
```

To type our equations into the calculator, we'll hit the \( X,T,\theta,n \) key when we want a \( t \) in the equation. Your equations should look like this:

```
Plot1 Plot2 Plot3
\( x_1t = 8t \)
\( y_1t = -16t^2 + 3t + 4 \)
\( x_2t = \)
\( y_2t = \)
\( x_3t = \)
\( y_3t = \)
```

To look at the numerical values for \( x(t) \) and \( y(t) \), hit the TBLSET key (2\(^{nd}\) WINDOW). Use the following settings:

```
TABLE SETUP
Tb1Start=0
\( \Delta Tb1=.1 \)
Indpt= Auto Ask
Depend= Auto Ask
```
These settings indicate that we would like to begin calculating values for \(x(t)\) and \(y(t)\) and time \(t = 0\). The deltaTbl value gives the time increment, so that the calculator will calculate value of \(x(t)\) and \(y(t)\) for \(t = 0.1, 0.2 \ldots \) etc.

Now hit the TABLE key (2nd GRAPH) and you should see the following:

<table>
<thead>
<tr>
<th>(T)</th>
<th>(x(t))</th>
<th>(y(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>0.1</td>
<td>1.8</td>
<td>4.14</td>
</tr>
<tr>
<td>0.2</td>
<td>3.2</td>
<td>4.15</td>
</tr>
<tr>
<td>0.3</td>
<td>4.8</td>
<td>4.14</td>
</tr>
<tr>
<td>0.4</td>
<td>6.2</td>
<td>4.15</td>
</tr>
<tr>
<td>0.5</td>
<td>4.8</td>
<td>4.14</td>
</tr>
</tbody>
</table>

Have the students think carefully about these values and that they represent the horizontal and vertical position of the ring at time \(t\).

To graph these equations, we need to first choose a good viewing window. Hit the Window key and set your variables as follows:

```
WINDOW
Tmin=0
Tmax=5
Tstep=.1
Xmin=0
Xmax=5
Xscl=1
Ymin=0
Ymax=6
Yscl=1
```

Scroll down and then set your remaining variables as follows:

```
WINDOW
Tstep=.1
Xmin=0
Xmax=5
Xscl=1
Ymin=0
Ymax=6
Yscl=1
```

Notice we have some “new” things to consider when we graph a parametric equation. We set the Tstep to 0.1 since we want the calculator to graph the position of the ring for each 1/10 of a second. The Tmin = 0 because we start at time \(t = 0\) and the Tmax = 5 because we want to plot the position of the ring for 5 seconds. The Xmin, Xmax, Xscl, Ymin, Ymax, and Yscl were chosen based on the table values.

Now hit graph and you should see the following:
If you hit the TRACE key and scroll left and right, you will see the \( x \)-value, \( y \)-value and \( t \)-value at each point.

In order to find out whether or not Sam hits his target, we need to calculate the position of the top of the target peg. The game is 4 feet from Sam and the game is on the one foot stool. The peg measures 2 inches and is located 4 inches from the bottom of the game. This information indicates that the top of the peg should be at the coordinates (4, 1.5).

If we trace on our graph to the point where \( x = 4 \), we see that the \( y \)-value is 1.5 when \( t = 0.5 \), so Sam has hit his mark again!
Introduction to Parametric Equations – Ring Toss Problem  
Advanced Functions and Modeling  
Student Handout

Each year Sam looks forward to the ring toss at the county fair. Sam is not happy to hit just any peg, but instead he prides himself on hitting the middle peg in the front row each year. Suppose Sam tosses the ring with an initial horizontal velocity of 8 feet/sec and an initial vertical velocity of 3 feet/sec and that he’ll release the ring when the ring is 4 feet above the ground. Sam lines up directly in front of the middle ring at a distance of 4 feet from the ring platform. Our job is to model the motion of the ring using parametric equations and find out whether or not Sam hits his target – the middle peg. The Ring Toss game is shown below. It sits on a short stool measuring 1 foot in height. The pegs are two inches high and the bottom of the first row of pegs is 4 inches from the bottom of the game.

We will think about the horizontal and vertical positions of the ring separately and write functions for each. These functions will be written as functions of time. Let \( x(t) \) represent the horizontal position of the ring at time \( t \) and \( y(t) \) represent the vertical position of the ring at time \( t \), where \( t \) is measured in seconds and the positions are measured in feet. These equations are called parametric equations and \( t \) is called the parameter.

Consider the horizontal position first. Since we know that Sam tosses the ring with a horizontal velocity of 8 ft/sec, our function is \( x(t) = 8t \).

Now for the vertical position, we know that the vertical position of the ring will be given by \( y(t) = -\frac{1}{2}gt^2 + v_0t + y_0 \), where \( g \) is the acceleration due to gravity (32 \( \text{ft/sec}^2 \)), \( v_0 \) is the initial velocity and \( y_0 \) is the initial height of the ring when Sam releases it. So we have, \( y(t) = -16t^2 + 3t + 4 \).
1. Using your calculator in parametric mode, type these equations and generate a table of values for the horizontal and vertical position of the ring for 0.7 seconds. Use a step size of 0.1.

Record those values in the table below:

<table>
<thead>
<tr>
<th>t</th>
<th>x(t)</th>
<th>y(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>0.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Sketch a graph of the ring’s position for the reasonable values in your table. Label your axes carefully. Label the points at time t = 0, t = 0.3 and t = 0.5.

3. Using your parametric equations, determine whether or not Sam will hit his target peg. Explain your reasoning.
Parametric Equations – Mars Rover Problem: Teacher Notes
Advanced Functions and Modeling

Goal: Optional Functions for Additional or Continued Study

Equipment and materials needed for students:
1. Copy of handout, 1 Student Handout for each student.
2. Graphing calculator
3. Paper and pencil for note taking.

This problem is designed to give students an opportunity to write parametric equations. The students will use the calculator to generate numerical values and graph parametric equations to model the motion of the Mars Rover - Spirit.

The Mars Rover Spirit landed safely on the surface of Mars in January 2004. The scientists were extremely happy with the landing site because it offers fairly level terrain. The landing site was close to the crater Gusev Crater. For more information on Spirit’s landing see the following web site:
http://marsrovers.jpl.nasa.gov/spotlight/navTarget01.html

At the time one source predicted that Spirit’s first destination would be a valley called Sleepy Hollow.

“"Sleepy Hollow," a shallow depression in the Mars ground near NASA's Spirit rover, may become an early destination when the rover drives off its lander platform in a week or so..’’ See the websites for more information.
http://www.hypography.com/article.cfm?id=34150

Refer to the map of Mars’ surface below. Suppose we set up a coordinate system so that the bottom left corner of the map is the point (0,0), and we begin observing Spirit’s path. After 20 hours we notice that Spirit is at the point (30,1) and then after 40 hours we notice that Spirit is at the point (25,10). These coordinates are measured in meters.
a. Assuming that Spirit moves at a constant rate, write parametric equations to represent Spirit’s path. For simplification we will assume that the surface of Mars is flat in Spirit’s vicinity. Sketch a graph of Spirit’s path for time \( t = 20 \) hours to \( t = 80 \) hours.

To write these equations, we will think about the vertical motion (how Spirit’s position South to North on the map changes) and the horizontal motion (how Spirit’s position West to East on the map changes) separately. Since we are assuming that Spirit moves at a constant rate, the horizontal position will be given by the linear function 

\[ x(t) = mt + b, \]

where \( m \) and \( b \) are constants.

At \( t = 20 \), Spirit is at \( x = 30 \), so we can write \( x(20) = 30 \). Then at time \( t = 40 \), Spirit is at \( x = 25 \), so we can write \( x(40) = 25 \). Substituting the values into the linear equation above, we have

\[
\begin{align*}
20m + b &= 30 \\
40m + b &= 25 
\end{align*}
\]

Solving this system of linear equations for \( m \) and \( b \) we get \( m = \frac{-1}{4} \) and \( b = 35 \).

So we have the horizontal position of the rover is

\[ x(t) = \frac{-1}{4} t + 35 \]

Using a similar method we can find the vertical motion along the path, \( y(t) = mt + b \).

\( y(20) = 1 \) and \( y(40) = 10 \) gives us

\[
\begin{align*}
20m + b &= 1 \\
40m + b &= 10 
\end{align*}
\]

Solving for \( m \) and \( b \) yields \( m = \frac{9}{20} \) and \( b = -8 \).

So \( y(t) = \frac{9}{20} t - 8 \).
Enter these parametric equations in the calculator, with a window of Tmin = 20, Tmax = 80, Tstep = 0.5, Xmin = -10, Xmax = 50, Xscl = 5, Ymin = -20, Ymax = 50, Yscl = 10.
You should see the graph below.

b. Suppose that the approximate center of Sleepy Hollow is (29,14). If spirit continues on the path defined above, how close will Spirit get to the center of Sleepy Hollow?

To find the solution to this problem we will write the distance from Spirit to Sleepy Hollow using the distance formula.

\[ d = \sqrt{(x(t) - 29)^2 + (y(t) - 14)^2} \]

We can graph this distance in parametric mode by entering the following in Y=:

\[
\begin{align*}
X_1T &= (-1/4)T + 35 \\
Y_1T &= (9/20)T - 8 \\
X_2T &= (t - 29)^2 \\
Y_2T &= (Y_1T - 14)^2 \\
X_3T &= \\
Y_3T &=
\end{align*}
\]

To get the variables X_1T, press VARS, arrow over to Y-VARS, arrow down to Parametric and choose the variable from the list.

The graph of the distance from Spirit to Sleepy Hollow is shown below with Xmax = 100.

Scroll to the approximate minimum value. You will find that the minimum distance is 5.5 meters and that minimum occurs when \( t = 44.5 \) hours.
Parametric Equations – Mars Rover Problem: Student Handout
Advanced Functions and Modeling

The Mars Rover Spirit landed safely on the surface of Mars in January 2004. The scientists were extremely happy with the landing site because it offers fairly level terrain. The landing site was close to the crater Gusev Crater. For more information on Spirit’s landing see the following web site:
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